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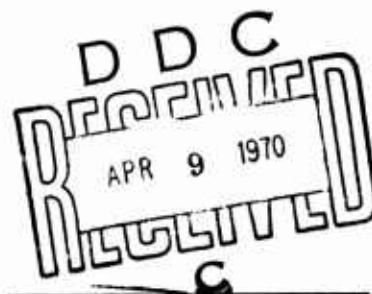
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**RADAR AND SONAR SIGNAL DESIGN:
USE OF STATIONARY PHASE
APPROXIMATION**

by
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March 1970

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The work described here was motivated by the continuing need for radar and sonar modulations that yield better ambiguity functions. It was performed at the Naval Undersea Research and Development Center, Pasadena Laboratory, during the period from October 1967 through October 1968 under Naval Ordnance Systems Command Task Assignment No. ORD-541-424/050-1/UF 17-341-404.

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PROBLEM

Apply the method of stationary phase to the integral defining Woodward's time-frequency correlation function $X(\tau, \beta)$ and investigate the usefulness of a particular elaboration of this approximation method as a tool for attacking the signal design problem, the latter being the finding of such amplitude $a(t)$ and phase $\phi(t)$ modulations that a prescribed ambiguity function $|X(\tau, \beta)|^2$ is obtained.

RESULTS

The method is heuristic and is applicable to signals such that amplitude $a(t)$ varies slowly relative to phase $\phi(t)$ over most of the signal duration, i. e., to phase-modulated signals. Those regions of the τ - β (range-Doppler) plane—except for a small neighborhood of the origin—on which the ambiguity function is relatively large are easily found. It is possible to see that by introducing finite discontinuities in the instantaneous frequency $\phi'(t)$, some of the ambiguity volume (the total volume under $|X(\tau, \beta)|^2$ is fixed) can be displaced outward and beyond physically interesting values of β . Further, the undisplaced volume can be distributed in azimuth about the origin. Hence, it seems possible to approximate the ideal ambiguity function—sharp peak at the origin, small values elsewhere—in this way.

RECOMMENDATIONS

The phase modulations arrived at in the later sections of this report, as well as any others obtained in the same way, are best regarded as constituents of candidate signals. Additional computation—digital, analog, or hand (using Erdelyi's asymptotic series)—is required in order to obtain a more accurate representation of the ambiguity function. Consideration of the accuracy which may be attained, and of the time and effort required, indicate that the first two are to be preferred over the third.

1. INTRODUCTION

Beginning with Woodward (Ref. 1), the time-frequency correlation function

$$\chi(\tau, \beta) = \int_{-\infty}^{\infty} u(t) \bar{u}(t + \tau) e^{-2\pi i \beta t} dt \quad (1.1)$$

has played an important role in the analysis of echo-ranging systems (Ref. 2 to 7). Much has been written about this function; Ref. 2 to 4 are concerned with its properties, its relevance, and its interpretation. The symbols in Eq. 1.1 have the following meanings: the variable of integration t has the dimensions of time, τ is related to the range, β is related to the range rate, $u(t)$ is the complex modulation, $i = \sqrt{-1}$, and the overscore denotes complex conjugation. The appearance of $u(t)$ in Eq. 1.1 results from writing the physical transmitted signal $s(t)$ as

$$s(t') = \text{Re}[u(t') e^{2\pi i f_c t'}] \quad (1.2)$$

where f_c is the carrier frequency and t' is the time, and from treating the target as a point scatterer.

The ambiguity function, $|\chi(\tau, \beta)|^2$, has more physical significance than $\chi(\tau, \beta)$, and the problem presented here is the one of choosing the modulation $u(t)$ so that an acceptable surface, $|\chi(\tau, \beta)|^2$ versus τ and β , is obtained. The qualitatively ideal surface is usually taken to be a $|\chi(\tau, \beta)|^2$ which is sharply peaked at the origin and relatively small elsewhere. Aside from the "random" phase-reversal signals (Ref. 3, 8) and the associated statistical interpretation of $|\chi(\tau, \beta)|^2$, it appears that a $u(t)$ which yields an ideal $|\chi(\tau, \beta)|^2$ has yet to be discovered. As might be expected, the lack of such an ideal modulation has not proved to be crucial. The tactical requirements of the system may admit some ambiguity (Ref. 9) owing to the presence of side lobes (additional peaks and ridges in the $|\chi(\tau, \beta)|^2$ surface). Or it may be possible to resolve the ambiguity (1) by observing targets over a sufficiently long period of time (Ref. 10) or (2) by transmitting more than one kind of signal (Ref. 4). On the other hand, it may be desirable or necessary to eliminate all ambiguities on each transmitted signal. In this last connection, we note that, since arbitrarily large values of $|\tau|$ and $|\beta|$ are not

expected to occur, it suffices to have $|\chi(\tau, \beta)|^2$ adequately small, except for the central peak, only on a rectangular domain of the $\tau - \beta$ plane bounded by $\pm|\tau|_m$ and $\pm|\beta|_m$. The particular elaboration of the method of stationary phase that is described below indicates how this latter sort of ambiguity function might be approximated in the case of a transmitted signal consisting of a single pulse, as opposed to a pulse burst or train (Ref. 4).

Although there is a theorem due to Siebert and to Wilcox (Ref. 2) by means of which $u(t)$ can, in principle, be calculated if $\chi(\tau, \beta)$ is prescribed, it appears that there is neither a theorem leading directly from $|\chi(\tau, \beta)|^2$ to $u(t)$ nor a theorem telling anything explicit about a possible relationship between the magnitude and the phase of $\chi(\tau, \beta)$. The signal design problem must therefore be attacked by devious means.

Let us recall that the method of stationary phase furnishes an approximate evaluation of the integral

$$F(\nu) = \int_a^c g(x) e^{i\nu f(x)} dx \quad (1.3)$$

where x , $f(x)$, and ν are real, and ν is large and positive. The relevance of the stationary phase approximation (SPA) to the signal design problem is due to the fact that the integral in Eq. 1.1, as well as the Fourier integrals for $u(t)$ and its spectrum $U(f)$, and the integral obtained by applying Parseval's formula to the right-hand member of Eq. 1.1 can be cast into the form of the integral in Eq. 1.3. In the case of Eq. 1.1, upon writing the complex modulation $u(t)$ in terms of the amplitude modulation $a(t)$ and the phase modulation $\phi(t)$,

$$u(t) = a(t) e^{2\pi i \phi(t)} \quad (1.4)$$

we obtain

$$\chi(\tau, \beta) = \int_{t_a}^{t_c} A(t; \tau) e^{2\pi i \Phi(t; \tau, \beta)} dt \quad (1.5)$$

where

$$A(t; \tau) = a(t) a(t + \tau) \quad (1.6)$$

$$\Phi(t; \tau, \beta) = \phi(t) - \phi(t + \tau) - \beta t \quad (1.7)$$

and we have changed to finite limits to agree with the fact that we shall be dealing with time-limited modulations:

$$a(t) = 0; t < t_1, t > t_2 \quad (1.8)$$

The SPA also yields some simple, auxiliary results that are useful in signal design. It is in this last respect that the application of the method of stationary phase described and illustrated here differs from previous applications (Ref. 2 to 4, and 11 to 14).

After a brief review of the SPA itself (Section 2), this report develops (Section 3) inequalities involving τ and β which, to within the SPA, specify the domains of the $\tau - \beta$ plane on which $|\chi(\tau, \beta)|^2$ is large.¹ We call such a domain a side lobe, since the domain corresponding to the central peak of $|\chi(\tau, \beta)|^2$ is excluded from these calculations. It is then ascertained (Section 4), by means of specific examples and some heuristic considerations, the kind of phase modulation $\phi(t)$ that results in (1) the displacement of some side lobes outward along the β -axis so that they lie beyond $\pm|\beta|_m$, and (2) the distribution in azimuth of those that are contiguous to the origin. Finally, the results are summarized and discussed in Section 5.

2. THE STATIONARY PHASE APPROXIMATION

The SPA is applicable to integrals such as are given by Eq. 1.3

$$F(\nu) = \int_a^c g(x) e^{i\nu f(x)} dx \quad (2.1)$$

where $f(x)$ and x are real and ν is large and positive. This approximation has been used frequently and apparently successfully in the analysis of physical problems. It is discussed in many places, usually as the method of stationary phase. Most of the material that follows comes from the monographs by Copson (Ref. 15) and Erdelyi (Ref. 16). The basic idea is as follows: because of the oscillating exponential, neighboring contributions of the integrand tend to cancel except in the vicinity of a point x_j at which the phase $\nu f(x)$ is stationary [$f'(x_j) = 0$] and at the end points.

Copson assumes that $f(z)$ and $g(z)$ are analytic functions of z (complex) in a simply connected, open region D containing the interval I ,

¹The possibility of doing this has also been noted by Vakman (Ref. 3), in connection with cubic phase modulation, $\phi(t) \propto t^3$.

$a \leq x \leq c$. It then follows that $f'(x)$ has a finite number of zeros on I ; these are the stationary points of the phase $\nu f(x)$. The interval I is presumed to be divided into a finite number of subintervals, in each of which $f'(x)$ does not vanish, or $f'(x)$ vanishes only at the left-hand end point, or $f'(x)$ vanishes only at the right-hand end point. These three cases are treated separately, and the results are as follows:

If $f(x)$ has no stationary point in $a \leq x \leq \kappa$, then

$$G(\nu) = \int_a^\kappa g(x) e^{i\nu f(x)} dx = \frac{g(\kappa)}{i\nu f'(\kappa)} e^{i\nu f(\kappa)} - \frac{g(a)}{i\nu f'(a)} e^{i\nu f(a)} + O\left(\frac{1}{\nu^2}\right) \quad (2.2)$$

as $\nu \rightarrow \infty$.

If $f(x)$ has one stationary point in $a \leq x \leq \kappa$, namely at $x = a$, then

$$\begin{aligned} G(\nu) &= \int_a^\kappa g(x) e^{i\nu f(x)} dx \\ &= \left[\frac{\pi}{\pm 2\nu f''(a)} \right]^{1/2} g(a) e^{i\nu f(a) \pm \pi i/4} + O\left(\frac{1}{\nu}\right) \end{aligned} \quad (2.3)$$

as $\nu \rightarrow \infty$, where the upper (lower) signs apply when $f''(a)$ is positive (negative).

If $f(x)$ has one stationary point in $a \leq x \leq \kappa$, namely at $x = \kappa$, then

$$\begin{aligned} G(\nu) &= \int_a^\kappa g(x) e^{i\nu f(x)} dx \\ &= \left[\frac{\pi}{\pm 2\nu f''(\kappa)} \right]^{1/2} g(\kappa) e^{i\nu f(\kappa) \pm \pi i/4} + O\left(\frac{1}{\nu}\right) \end{aligned} \quad (2.4)$$

as $\nu \rightarrow \infty$, where the upper (lower) signs apply when $f''(\kappa)$ is positive (negative).

Although these results will suffice for most of the discussion in the next two sections, a more general theorem is useful (1) for those cases where either $f''(a) = 0$ or $f''(\kappa) = 0$; (2) for establishing that the right-hand members of Eq. 2.2 through 2.4 are, in fact, the dominant terms of an asymptotic expansion; and (3) for obtaining more accurate asymptotic approximations. This theorem is derived by Erdelyi and includes stationary points of a more general character. It is assumed that the number of these more general stationary points in the interval I is finite, and that I is broken up as before except that, for definiteness,

$f(x)$ is monotonic increasing² on $a \leq x \leq \kappa$. The theorem reads as follows:

If $\lambda > 0$, $\mu \geq 1$; $g(x)$ is N times continuously differentiable for $a \leq x \leq \kappa$; $f(x)$ is differentiable and

$$f'(x) = (x - \alpha)^{\rho-1} (\kappa - x)^{\sigma-1} f_1(x) \quad (2.5)$$

where $\rho, \sigma \geq 1$, and $f_1(x)$ is positive and N times continuously differentiable for $a < x < \kappa$; then

$$\int_a^\kappa g(x) e^{i\nu f(x)} (x - \alpha)^{\lambda-1} (\kappa - x)^{\mu-1} dx = B(\nu) - A(\nu) \quad (2.6)$$

where

$$A(\nu) \sim - \sum_{n=0}^{N-1} \frac{k^{(n)}(0)}{n! \rho} \Gamma\left(\frac{n+\lambda}{\rho}\right) e^{\pi i(n+\lambda)/2\rho} \nu^{-(n+\lambda)/\rho} e^{i\nu f(a)} \quad (2.7)$$

and

$$B(\nu) \sim - \sum_{n=0}^{N-1} \frac{\ell^{(n)}(0)}{n! \sigma} \Gamma\left(\frac{n+\mu}{\sigma}\right) e^{-\pi i(n+\mu)/2\sigma} \nu^{-(n+\mu)/\sigma} e^{i\nu f(\kappa)} \quad (2.8)$$

as $\nu \rightarrow \infty$, with $k(u)$ given by

$$\left. \begin{aligned} k(u) &= g_1(x) u^{1-\lambda} \frac{dx}{du} \\ u^\rho &= f(x) - f(\alpha) \\ g_1(x) &= g(x) (x - \alpha)^{\lambda-1} (\kappa - x)^{\mu-1} \end{aligned} \right\} \quad (2.9)$$

and $\ell(v)$ given by

$$\left. \begin{aligned} \ell(v) &= g_1(x) v^{1-\mu} \frac{dx}{dv} \\ v^\sigma &= f(\kappa) - f(x) \end{aligned} \right\} \quad (2.10)$$

The notation

$$A(\nu) \sim \sum_{n=0}^{N-1} a_n \phi_n(\nu) \text{ as } \nu \rightarrow \infty$$

² If $f(x)$ is monotonic decreasing, replace i with $-i$ and $f(x)$ with $-f(x)$.

means that

$$A(\nu) = \sum_{n=0}^{N-1} a_n \phi_n(\nu) + o(\phi_{N-1}) \text{ as } \nu \rightarrow \infty$$

The presence or absence of a stationary point within the interval of integration is manifested in the values of ρ and σ . If there is no stationary point, $\rho = \sigma = 1$; otherwise, either ρ or σ , or both, is greater than unity.

3. GENERAL CONSIDERATIONS

The use of the SPA on Eq. 1.5 is now considered in more detail. No loss in generality results from choosing the time origin so that

$$t_1 = -\frac{T}{2}, t_2 = +\frac{T}{2} \quad (3.1)$$

We then have

$$a(t) = 0 \text{ for } t < -\frac{T}{2}, t > \frac{T}{2} \quad (3.2)$$

from Eq. 1.8. Also, because of the symmetry condition,

$$|\chi(-\tau, -\beta)| = |\chi(\tau, \beta)| \quad (3.3)$$

we may take

$$\tau > 0 \quad (3.4)$$

throughout. Equation 1.5 then becomes

$$\chi(\tau, \beta) = \int_{-T/2}^{(T/2)-\tau} A(t; \tau) e^{2\pi i \Phi(t; \tau, \beta)} dt \quad (3.5)$$

where

$$\Phi(t; \tau, \beta) = \phi(t) - \phi(t + \tau) - \beta t \quad (3.6)$$

and

$$A(t; \tau) = a(t)a(t + \tau) \quad (3.7)$$

We note that

$$\chi(\tau, \beta) = 0 \text{ for } \tau \geq T \quad (3.8)$$

The simplest case obtains when $\phi(z)$ and $a(z)$ are analytic functions of z ($\text{Re } z = t$) on a simply connected, open region D containing the t -interval $[-T/2, T/2]$. Since this assumption conflicts with Eq. 3.2, we have to suppose, for the purpose of applying the first three theorems given in the previous section, that $a(t)$ is analytically continued to the left beyond $-T/2$ and to the right beyond $+T/2$. The theorems of the previous section then tell us that the SPA will be a good approximation if, aside from stationary points, the phase $2\pi\Phi(t; \tau, \beta)$ varies rapidly with t compared to $A(t; \tau)$ on $[-T/2, T/2]$. We assume this to be true henceforth, and it then follows from Eq. 3.6 and 3.7 that the phase modulation $\phi(t)$ must vary relative to the amplitude modulation $a(t)$ on $[-T/2, T/2]$. In a specific case, a factor ν can be taken out of $2\pi\Phi(t; \tau, \beta)$. These theorems also tell us that the ambiguity function $|\chi(\tau, \beta)|^2$ will be larger by a factor ν when the stationary point(s) lie within the interval of integration; i. e., when

$$-\frac{T}{2} \leq t_j \leq \frac{T}{2} \quad (3.9)$$

with the stationary point t_j satisfying

$$\left[\frac{\partial \Phi(t; \tau, \beta)}{\partial t} \right]_{t=t_j} = [\phi'(t) - \phi'(t + \tau) - \beta]_{t=t_j} = 0 \quad (3.10)$$

Hence, t_j is a function of τ and β , and Eq. 3.9 may be written as

$$-\frac{T}{2} \leq t_j(\tau, \beta) \leq \frac{T}{2} - \tau \quad (3.11)$$

The equations

$$\left. \begin{aligned} t_j(\tau, \beta) &= -\frac{T}{2} \\ t_j(\tau, \beta) &= \frac{T}{2} - \tau \end{aligned} \right\} \quad (3.12)$$

define the loci, β versus τ , corresponding to a stationary point at the ends of the interval of integration $[-T/2, (T/2) - \tau]$. According to Eq. 3.10, these loci are given by

$$\beta = [\phi'(t) - \phi'(t + \tau)]_{t=-T/2} \equiv f_1(\tau) \quad (3.13)$$

and

$$\beta = [\phi'(t) - \phi'(t + \tau)]_{t=T/2-\tau} \equiv f_2(\tau) \quad (3.14)$$

We shall refer to $f_1(\tau)$ and $f_2(\tau)$ as the terminal curves. When the stationary point t_j is within the interval of integration, the loci are given by

$$\beta = [\phi'(t) - \phi'(t + \tau)]_{t=t_j}, \quad -\frac{T}{2} < t_j < \frac{T}{2} - \tau \quad (3.15)$$

from Eq. 3.10. It is obvious that all these loci³ pass through the origin, $\tau = \beta = 0$. They also intersect at $\tau = T$, as may be seen by observing that there is then only one possible value for t_j , namely $t_j = -T/2$. Collectively, these loci constitute the side lobe, the boundaries of which will be either certain members of the family of loci which enclose all the others or they will be envelope curves.

In this connection, we consider next the differential

$$d\beta = [\phi''(t) - \phi''(t + \tau)]_{t=t_j} dt_j = \left[\frac{\partial^2 \Phi(t; \tau, \beta)}{\partial t^2} \right]_{t=t_j} dt_j \quad (3.16)$$

which, for fixed τ , gives the displacement $d\beta$ involved in moving from one locus to another which is close by. By looking at the sign of $[\partial^2 \Phi(t; \tau, \beta)/\partial t^2]_{t=t_j}$ as the stationary point t_j moves into the interval of integration, we shall be able to see how the loci (Eq. 3.15) are disposed relative to the terminal curves $f_1(\tau)$ and $f_2(\tau)$. Clearly, the terminal curves and the comments relative to Eq. 3.15 apply to any other stationary points which may exist in addition to t_j . Hence, the extent of the side lobe may be determined in this way regardless of how many stationary points are present. However, the existence of more than one stationary point may influence the magnitude of $\chi(\tau, \beta)$ on the side lobe: by applying Eq. 2.3 and 2.4 to Eq. 3.5, we obtain

$$\chi(\tau, \beta) = \sum_j \left[\pm \frac{\partial^2 \Phi(t; \tau, \beta)}{\partial t^2} \right]_{t=t_j}^{-1/2} A(t_j; \tau) e^{i[2\pi \Phi(t_j; \tau, \beta) \pm \pi/4]} \quad (3.17)$$

with $t_j = t_j(\tau, \beta)$ a solution of Eq. 3.10. It also has the effect of making the SPA less accurate as $\tau \rightarrow T$ from below and the multiple stationary points coalesce (a circumstance which is not considered important, in view of Eq. 3.8).

The simplest situation obtains when $\partial^2 \Phi(t; \tau, \beta)/\partial t^2$ is independent of t and has the same sign on $0 < \tau < T$; say

$$\frac{\partial^2 \Phi(t; \tau, \beta)}{\partial t^2} > 0, \quad 0 < \tau < T \quad (3.18)$$

³The fact that a parametric family of curves is generated by varying the stationary point was noted by Sorkin (Ref. 13) but not developed in detail.

(We note in passing that $\partial^2 \Phi(t; \tau, \beta) / \partial t^2$ is always independent of β .) If, in addition,

$$f_1(\tau) < f_2(\tau), \quad 0 < \tau < T \quad (3.19)$$

then the side lobe is specified by

$$f_1(\tau) \leq \beta \leq f_2(\tau), \quad 0 < \tau < T \quad (3.20)$$

an inequality which is equivalent to Eq. 3.9 and which occurs in Example 1 in the next section.

The question arises whether the inequality sign could be reversed in either (3.18) or (3.19), for in that event there would be no side lobe at all, a desirable state of affairs. However, this would lead to the contradiction that $f_1(\tau)$ and $f_2(\tau)$ are not members of the same family of curves, there being no way of getting from one to the other by varying t_j . The possibility that one of the signs may be reversed over just $0 < \tau < T/2$, for example, may be disposed of on the same grounds.

The more complicated cases arise when $\partial^2 \Phi(t; \tau, \beta) / \partial t^2$ depends upon both t and τ . An enumeration and analysis of these has not been attempted here since the general procedure seems clear. Starting with $f_1(\tau)$ (the member of the family of loci corresponding to $t_j = -T/2$), the evolution of the remaining members is ascertained by repeated application of Eq. 3.16 as t_j moves from $-T/2$ to $T/2 - \tau$. If more convenient, it is possible to start with $f_2(\tau)$, in which case t_j evidently moves from $(T/2) - \tau$ to $-T/2$, and dt_j in Eq. 3.16 is negative. It is not uncommon for $\partial^2 \Phi(t; \tau, \beta) / \partial t^2$ to change sign, perhaps at \hat{t} , $-T/2 < \hat{t} < T/2 - \tau$. If Eq. 3.19 holds, and the change in sign is from plus to minus, then the side lobe will be given by

$$f_1(\tau) \leq \beta \leq \hat{f}(\tau), \quad 0 < \tau < T \quad (3.21)$$

where

$$\hat{f}(\tau) = [\phi'(t) - \phi'(t + \tau)]_{t=\hat{t}}$$

is the equation for the envelope curve (Ref. 17). Examples 2 and 3 below are instances of this sort.

Suppose now, contrary to the assumption following Eq. 3.8, that the amplitude $a(t)$ has finite discontinuities at $t = \pm T/2$. Such a case

is of interest as a mathematical idealization of certain physical waveforms. Let $h(t)$ be defined by

$$\begin{aligned} h(t) &= a(t) \text{ for } |t| < \frac{T}{2} \\ h(t) &= \text{analytic continuation of } a(t) \text{ for } |t| \geq \frac{T}{2} \end{aligned} \quad (3.22)$$

Then, by using

$$A(t; \tau) = h(t)h(t + \tau) \quad (3.23)$$

in place of Eq. 3.7, the previous results are again obtained.

When there are finite discontinuities in $a'(t)$, $\phi(t)$, $\phi'(t)$, etc., the interval of integration $[-T/2, (T/2) - \tau]$ is partitioned so that the points of discontinuity coincide with the limits of the integrals making up Eq. 3.5, and then analytic continuations of $a(t)$ and $\phi(t)$ are introduced in order that the theorems of Section 2 can be applied to each of the constituent integrals. Of course, the analytically continued functions have a strictly formal role; all that needs to be done in a calculation is to avoid integrating through a discontinuity. Results similar to Eq. 3.11, 3.13, 3.14, 3.15, and 3.16 then apply for each segment of the original interval $[-T/2, (T/2) - \tau]$.

In the case of Erdelyi's theorem (Eq. 2.6 through 2.8), neither $a(t)$ nor $\phi(t)$ need be analytic. It has not been possible to see whether this weaker hypothesis has any desirable consequences insofar as the signal design problem is concerned, except that the theorem gives a finite result in some cases in which Eq. 2.3 and 2.4 do not. In particular, on the β -axis, $\partial^2 \Phi(t; \tau, \beta) / \partial t^2$ vanishes identically, and

$$\begin{aligned} \chi(\tau, \beta) &\rightarrow \infty \\ \beta &\rightarrow 0 \end{aligned}$$

according to Eq. 2.3 and 2.4, while Erdelyi's theorem gives a finite result, which, for the case of a rectangular $a(t)$, turns out to be exact. This circumstance is useful in a supplementary way in that degenerate side lobes can be defined (Eq. 4.6 and 4.7 below).

The SPA fails completely at the origin because $\Phi(t; 0, 0) \equiv 0$. But it is known from other considerations that $|\chi|^2$ attains a maximum there. Also, its behavior in a small neighborhood of the origin can be studied by means of a Taylor series expansion.

Thus, our use of the SPA, principally to map out those regions of the plane on which $|\chi(\tau, \beta)|^2$ is large, excludes the central maximum and a narrow strip about the β -axis. Along the β -axis, one can use either Erdelyi's theorem or the exact equations

$$\chi(\tau, \beta) = \int_{-\infty}^{\infty} [a(t)]^2 e^{-2\pi i \beta t} dt = \int_{-\infty}^{\infty} A(f) \overline{A(f+\beta)} df \quad (3.24)$$

which follow from Eq. 3.5 and Parseval's formula. Here, $A(f)$ is the spectrum of $a(t)$.

It is natural to inquire into the possibility of attacking the signal design problem synthetically within the present context, i. e., by specifying the bounding curves of the side lobes, then working toward the modulation, Eq. 1.4. It is easy to convince oneself that this procedure is ambiguous. In particular, the integrations with respect to t that are necessary lead to functions of τ which may be specified arbitrarily. Hence, it seems more satisfactory to work toward the placement of the side lobes by trial and error.

4. SOME EXAMPLES

In this section, the elaboration of the SPA described above is applied to several different phase modulations $\phi(t)$. The amplitude modulation $a(t)$ is unspecified, except that it is assumed to be such that the theorems of Section 2 are applicable, and to vary slowly compared to $\phi(t)$ over the duration T of the signal. The supposition that $|\chi(\tau, \beta)|^2$ is large on the side lobes (stationary points within the interval of integration) and small elsewhere (except for the central lobe) will then be admissible. For convenience, we introduce the function $\Theta(t; \tau)$:

$$\Theta(t; \tau) = \phi'(t) - \phi'(t + \tau) \quad (4.1)$$

The two common modulations,

$$\phi(t) = c \text{ (monotone)} \quad (4.2)$$

and

$$\phi(t) = \frac{1}{2} c_2 t^2 \text{ (linear FM)}^4 \quad (4.3)$$

⁴Frequency modulation.

where c and c_2 are constants, lead, respectively, to the equations

$$\frac{\partial \Phi(t; \tau, \beta)}{\partial t} = -\beta = 0 \quad (4.4)$$

and

$$\frac{\partial \Phi(t; \tau, \beta)}{\partial t} = -c_2 \tau - \beta = 0 \quad (4.5)$$

for the stationary point. Since t does not appear in these equations, there are no stationary points of the ordinary kind, and Eq. 2.3 and 2.4 do not apply. On the other hand, Erdelyi's theorem yields a finite result, which, it turns out, is exact for a rectangular $a(t)$. Now, in the case of a rectangular $a(t)$, $|\chi(\tau, \beta)|^2$ decreases in directions orthogonal to the lines

$$\beta = 0 \text{ (monotone)} \quad (4.6)$$

and

$$\beta = -c_2 \tau \text{ (linear FM)} \quad (4.7)$$

It is, therefore, not unreasonable to consider that these equations constitute a degenerate sort of side lobe.

Example 1.

A slightly more complicated phase modulation is

$$\phi(t) = \frac{1}{2} c_2 t^2 + \frac{1}{3} c_3 t^3 \quad (4.8)$$

which may be called linear plus quadratic FM. The term linear in t has been omitted because it has no effect on $|\chi(\tau, \beta)|^2$. We have

$$\begin{aligned} \Phi(t; \tau, \beta) = & \left[\frac{1}{2} c_2 t^2 + \frac{1}{3} c_3 t^3 \right] \\ & - \left[\frac{1}{2} c_2 (t + \tau)^2 + \frac{1}{3} c_3 (t + \tau)^3 \right] - \beta t \end{aligned} \quad (4.9)$$

$$\frac{\partial \Phi(t; \tau, \beta)}{\partial t} = -[c_3 \tau^2 + (c_2 + 2c_3 \tau) \tau + \beta] \quad (4.10)$$

$$\Theta(t; \tau) = -[c_3 \tau^2 + (c_2 + 2c_3 \tau) \tau] \quad (4.11)$$

and

$$\frac{\partial^2 \Phi(t; \tau, \beta)}{\partial t^2} = -2c_3 \tau \quad (4.12)$$

From the equation

$$\frac{\partial \Phi(t; \tau, \beta)}{\partial t} = 0$$

we find that there is just one stationary point t_j , given by

$$t_j = -\frac{1}{2c_3\tau} (c_3\tau^2 + c_2\tau + \beta) \quad (4.13)$$

The terminal curves $f_1(\tau)$ and $f_2(\tau)$ are given by

$$\begin{aligned} f_1(\tau) &= [\Theta(t_j; \tau)]_{t_j = -T/2} \\ &= -c_3 \left[\tau + \frac{1}{2} \left(\frac{c_2}{c_3} - T \right) \right]^2 + \frac{c_3}{4} \left(\frac{c_2}{c_3} - T \right)^2 \end{aligned} \quad (4.14)$$

and

$$\begin{aligned} f_2(\tau) &= [\Theta(t_j; \tau)]_{t_j = T/2 - \tau} \\ &= c_3 \left[\tau - \frac{1}{2} \left(\frac{c_2}{c_3} + T \right) \right]^2 - \frac{c_3}{4} \left(\frac{c_2}{c_3} + T \right)^2 \end{aligned} \quad (4.15)$$

and the remaining members of the family constituting the side lobe by

$$\beta = -c_3\tau^2 - (c_2 + 2c_3t_j)\tau, \quad -\frac{T}{2} < t_j < \frac{T}{2} - \tau \quad (4.16)$$

Suppose, for definiteness, that

$$c_3 < 0, \quad \frac{c_2}{c_3} - T > 0 \quad (4.17)$$

The loci of the terminal curves then are as sketched in Fig. 1. On the interval $0 < \tau < T$, $f_1(\tau) < f_2(\tau)$, and from Eq. 4.12 and 4.17, $\partial^2 \Phi(t; \tau, \beta) / \partial t^2 > 0$. If Eq. 3.16 is now presumed to be applied repeatedly, starting from $t_j = -T/2$ and the curve $f_1(\tau)$, we can see that the inequality (3.20) applies, and that the side lobe of $|\chi(\tau, \beta)|^2$ for $\tau > 0$ is as indicated by the shaded region. The loci (4.16) lie within this region on the interval $0 < \tau < T$.

For convenience, such a diagram is called an ambiguity diagram here. The part for $\tau < 0$ is obtained by reflecting the shaded region in the origin (see Eq. 3.3). The central lobe is not shown.

Example 2.

In this example, we take

$$\varphi(t) = \frac{1}{4} ct^4 \quad (4.18)$$

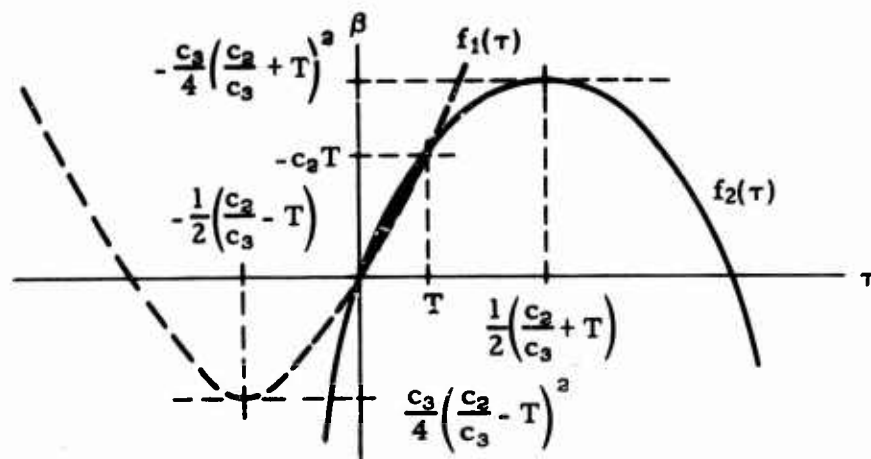


FIG. 1. Ambiguity Diagram for Linear Plus Quadratic FM, Eq. 4.8; $\tau > 0$. $|x(\tau, \beta)|^2$ is large on the shaded region.

which may be called cubic FM, and for which the following results are obtained:

$$\Phi(t; \tau, \beta) = \frac{1}{4} ct^4 - \frac{1}{4} c(t + \tau)^4 - \beta t \quad (4.19)$$

$$\frac{\partial \Phi(t; \tau, \beta)}{\partial t} = -c\tau(3t^2 + 3t\tau + \tau^2) - \beta \quad (4.20)$$

$$\Theta(t; \tau) = -c\tau(3t^2 + 3t\tau + \tau^2) \quad (4.21)$$

and

$$\frac{\partial^2 \Phi(t; \tau, \beta)}{\partial t^2} = -3c\tau(2t + \tau) \quad (4.22)$$

From Eq. 4.20 it is seen that there are two stationary points,

$$t_j = -\frac{\tau}{2} \pm \frac{1}{6} \sqrt{-3\left(\tau^2 + \frac{4\beta}{c\tau}\right)} \quad (4.23)$$

The terminal curves are

$$f_1(\tau) = [\Theta(t_j; \tau)]_{t_j = -\tau/2} = -c\tau \left[\left(\tau - \frac{3T}{4} \right)^2 + 3 \left(\frac{T}{4} \right)^2 \right] \quad (4.24)$$

$$f_2(\tau) = [\Theta(t_j; \tau)]_{t_j = T/2 - \tau} = -c\tau \left[\left(\tau - \frac{3T}{4} \right)^2 + 3 \left(\frac{T}{4} \right)^2 \right] \quad (4.25)$$

Since the loci of $f_1(\tau)$ and $f_2(\tau)$ coincide and $\partial^2\Phi(t; \tau, \beta)/\partial t^2$ depends upon t , the ambiguity diagram for this example will have a character entirely different from that of the previous case. For use in Eq. 3.16,

$$\left[\frac{\partial^2\Phi(t; \tau, \beta)}{\partial t_j^2} \right]_{t_j = -T/2} = 3c\tau(T - \tau) \quad (4.26)$$

$$\left[\frac{\partial^2\Phi(t; \tau, \beta)}{\partial t_j^2} \right]_{t_j = T/2 - \tau} = -3c\tau(T - \tau) \quad (4.27)$$

Taking

$$c > 0 \quad (4.28)$$

and referring to Eq. 3.16 and 4.26, it is found that, as a stationary point moves from $-T/2$ into the interval $[-T/2, (T/2) - \tau]$, the loci

$$\beta = [\Theta(t; \tau)]_{t=t_j} = -c\tau(3t_j^2 + 3t_j\tau + \tau^2), \quad -\frac{T}{2} < t_j < \frac{T}{2} - \tau \quad (4.29)$$

are displaced upward from $f_1(\tau)$. Similarly, Eq. 3.16 and 4.27 reveal that, as a stationary point moves from $(T/2) - \tau$ into the interval $[-T/2, (T/2) - \tau]$, the loci (4.29) are, again, displaced upward from $f_1(\tau) [= f_2(\tau)]$. It is therefore necessary to determine the envelope curve $\hat{f}(\tau)$ corresponding to the solution \hat{t} of

$$\frac{\partial^2\Phi(t; \tau, \beta)}{\partial t^2} = -3c\tau(2t + \tau) = 0 \quad (4.30)$$

By inspection, this solution is

$$\hat{t} = -\frac{\tau}{2} \quad (4.31)$$

from which

$$\hat{f}(\tau) = [\Theta(t; \tau)]_{t=\hat{t}} = -\frac{c}{4}\tau^3 \quad (4.32)$$

and

$$\left[\frac{\partial^3\Phi(t; \tau, \beta)}{\partial t^3} \right]_{t=\hat{t}} = -6c\tau < 0 \quad (4.33)$$

It follows that the side lobe is specified by

$$f_1(\tau) \leq \beta \leq \hat{f}(\tau), \quad 0 < \tau < T \quad (4.34)$$

The ambiguity diagram for $\tau > 0$ and with the central lobe excluded appears in Fig. 2.

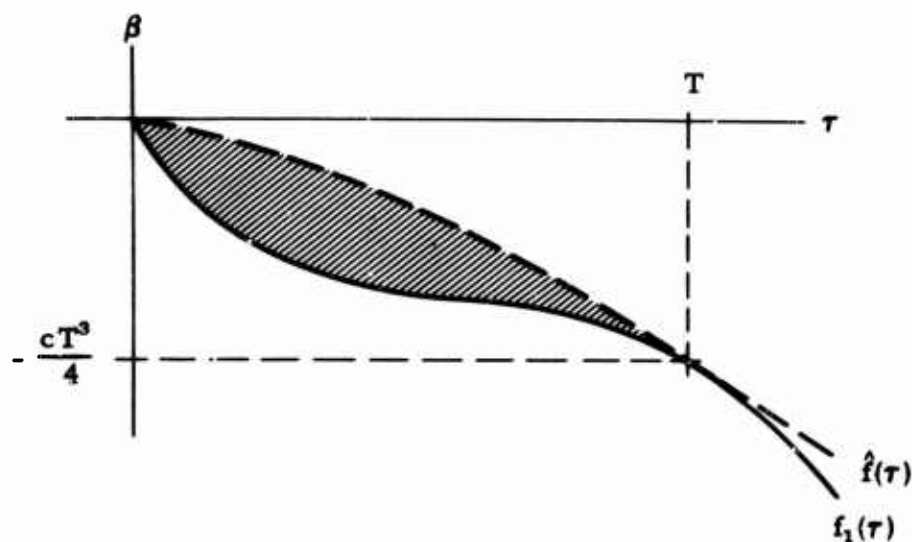


FIG. 2. Ambiguity Diagram for the Modulation of Eq. 4.18, Cubic FM.

It may be worth noting that explicit use was not made of the expression for the stationary points given in Eq. 4.23. That equation is included only for completeness; were $|\chi(\tau, \beta)|^2$ to be calculated on the side lobe, it would be needed for substitution into Eq. 3.17. Similar comments apply to Eq. 4.13 in the preceding example.

Example 3.

Because the case of sinusoidal FM

$$\phi(t) = c_1 \sin c_2 t$$

is somewhat awkward to discuss for arbitrary values of c_2 , a specific and convenient value is assumed:

$$\phi(t) = c_1 \sin 2\pi \frac{t}{T} \quad (4.35)$$

From this there follows

$$\Phi(t; \tau, \beta) = c_1 \sin 2\pi \frac{t}{T} - c_1 \sin 2\pi \frac{t + \tau}{T} - \beta t \quad (4.36)$$

$$\frac{\partial \Phi(t; \tau, \beta)}{\partial t} = \frac{4\pi c_1}{T} \sin \pi \frac{\tau}{T} \sin 2\pi \frac{t + \frac{\tau}{2}}{T} - \beta \quad (4.37)$$

$$\Theta(t; \tau) = \frac{4\pi c_1}{T} \sin \pi \frac{\tau}{T} \sin 2\pi \frac{t + \frac{\tau}{2}}{T} \quad (4.38)$$

$$f_1(\tau) = [\Theta(t_j; \tau)]_{t_j = -T/2}$$

$$= -\frac{2\pi c_1}{T} \left[1 + \cos 2\pi \left(\frac{\tau}{T} - \frac{1}{2} \right) \right] \quad (4.39)$$

$$f_2(\tau) = [\Theta(t_j; \tau)]_{t_j = T/2 - \tau} = -f_1(\tau) \quad (4.40)$$

$$\frac{\partial^2 \Phi(t; \tau, \beta)}{\partial t^2} = 8 \left(\frac{\pi}{T} \right)^2 c_1 \sin \pi \frac{\tau}{T} \cos 2\pi \frac{t + \frac{\tau}{2}}{T} \quad (4.41)$$

$$\left[\frac{\partial^2 \Phi(t_j; \tau, \beta)}{\partial t_j^2} \right]_{t_j = -T/2} = -c_1 \left(\frac{2\pi}{T} \right)^2 \sin 2\pi \frac{\tau}{T} \quad (4.42)$$

and

$$\left[\frac{\partial^2 \Phi(t_j; \tau, \beta)}{\partial t_j^2} \right]_{t_j = T/2 - \tau} = \left[\frac{\partial^2 \Phi(t_j; \tau, \beta)}{\partial t_j^2} \right]_{t_j = -T/2} \quad (4.43)$$

Since the equation for the stationary points, $\partial \Phi(t; \tau, \beta) / \partial t = 0$, is transcendental, the evaluation of the ambiguity function would be difficult in this case. The ambiguity diagram can, of course, be determined with less effort.

From Eq. 4.39, 4.40, 4.42, 4.43, and 3.16, we can construct the preliminary diagram shown in Fig. 3 where, for definiteness, we have taken

$$c_1 < 0 \quad (4.44)$$

The arrows in Fig. 3 indicate the directions in which neighboring members of the family of loci making up the side lobe are to be found. The equation for these loci is

$$\beta = [\Theta(t; \tau)]_{t=t_j} = \frac{4\pi c_1}{T} \sin \pi \frac{\tau}{T} \sin 2\pi \frac{t_j + \frac{\tau}{2}}{T}, \quad -\frac{T}{2} < t_j < \frac{T}{2} - \tau \quad (4.45)$$

The points P_1 and P_2 seem to have a sort of singular character, and we put $\tau = T/2$ in Eq. 4.45 to find out what happens there. This gives

$$\beta = \frac{4\pi c_1}{T} \cos 2\pi \frac{t_1}{T}, \quad -\frac{T}{2} < t_1 < 0 \quad (4.46)$$

which tells us that the point P_1 moves downward to P_2 as the parameter t_j moves through its allowed range.

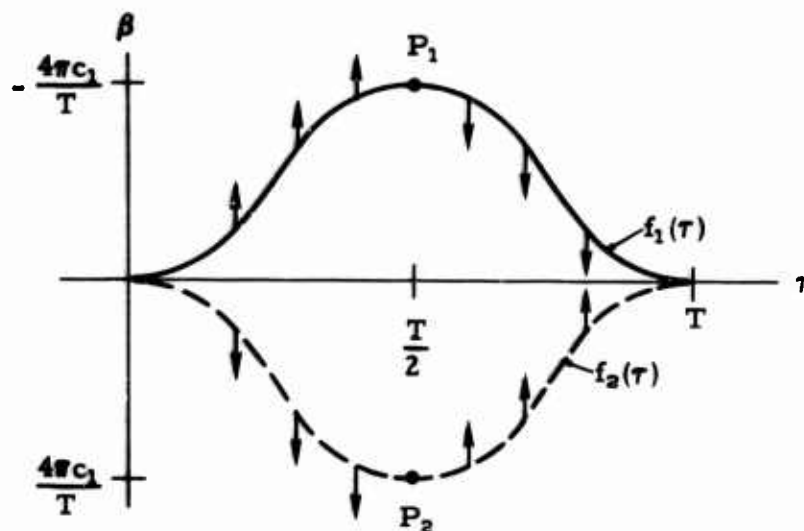


FIG. 3. Displacements From the Terminal Curves Corresponding to Small Increments in the Parameter t_j (Stationary Point); Sinusoidal FM of Eq. 4.35.

In order to elucidate matters further, we look for the values \hat{t} at which $\partial^2 \Phi(t; \tau, \beta) / \partial t^2$ vanishes. By so doing, the envelopes of the loci (4.45) can be determined. It is sufficient to look at the time-dependent factor in Eq. 4.41. This vanishes when

$$\hat{t} = \frac{1}{2} \left[(2m + 1) \frac{T}{2} - \tau \right], \quad m = \text{integer or zero} \quad (4.47)$$

By combining this with

$$-\frac{T}{2} < \hat{t} < \frac{T}{2} - \tau$$

then using $0 < \tau < T$ (Eq. 3.4 and 3.8), we arrive at

$$|2m + 1| < 2 \left(1 - \frac{\tau}{T} \right), \quad m = 0, \pm 1, \pm 2, \dots \quad (4.48)$$

This inequality has the solutions $m = 0$ or -1 for $0 < \tau < T/2$, and no solutions for $T/2 < \tau < T$. Upon using these results in Eq. 4.47, then substituting into Eq. 4.41, it is found that

$$\frac{\partial^2 \Phi(t; \tau, \beta)}{\partial t^2} = 0 \text{ at } \hat{t} = \pm \frac{T}{4} - \frac{\tau}{2}, \quad 0 < \tau < \frac{T}{2}$$

$$\frac{\partial^2 \Phi(t; \tau, \beta)}{\partial t^2} \neq 0, \quad \frac{T}{2} < \tau < T \quad (4.49)$$

Hence, for $T/2 < \tau < T$ the sense of the displacements shown in Fig. 3 persists throughout the allowed interval for t_j . For $0 < \tau < T/2$, the envelope curves are found by substituting $\hat{t} = \pm T/4 - \tau/2$ for t_j in Eq. 4.45:

$$\begin{aligned}\hat{f}_1(\tau) &= -\frac{4\pi C_1}{T} \sin \pi \frac{\tau}{T} \\ \hat{f}_2(\tau) &= -\hat{f}_1(\tau)\end{aligned}\quad (4.50)$$

The sense of the displacements for $0 < \tau < T/2$ shown in Fig. 3 reverses at these curves.

The results may now be combined so as to obtain the inequalities specifying the side lobe,

$$\begin{aligned}\hat{f}_2(\tau) \leq \beta \leq \hat{f}_1(\tau), \quad 0 < \tau \leq \frac{T}{2} \\ f_2(\tau) \leq \beta \leq f_1(\tau), \quad \frac{T}{2} \leq \tau \leq T\end{aligned}\quad (4.51)$$

and the corresponding ambiguity diagram, shown in Fig. 4. As before, this diagram is for $\tau > 0$; the complete diagram, except for the central lobe, may be obtained by inversion in the origin.

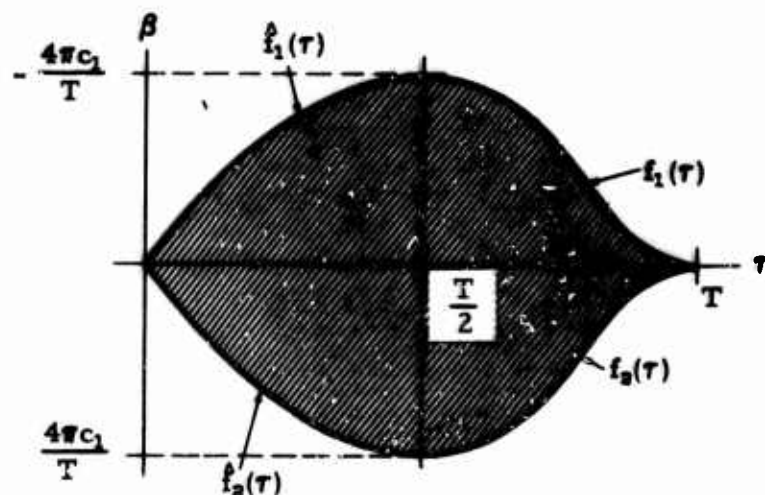


FIG. 4. Ambiguity Diagram for the Sinusoidal FM of Eq. 4.35.

In the example considered thus far, the phase modulation $\phi(t)$ satisfied the hypotheses of the theorems stated in Section 2, i. e., $\phi(z)$, with $\text{Re } z = t$, was analytic on an open, simply connected region

containing the t -interval $[-T/2, T/2]$. In each case, it was found that the ambiguity function $|\chi(\tau, \beta)|^2$ had side lobes connected to the origin and extending outward to $\tau = \pm T$. The discussion in Section 3 following the inequality (3.20) indicates that this sort of result will always be obtained.

We are thus led to consider modulations $\phi(t)$ which are not analytic as a possible means to obtain an ambiguity function all or part of whose side lobes are displaced outward beyond the domain of values of τ and β which are expected to occur in practice. As pointed out in Section 1, such a $|\chi(\tau, \beta)|^2$ would be a desirable approximation to the ideal.

Example 4.

To this end, let us consider the modulations given by

$$a(t) = 0, \quad t < 0, \quad t > T \quad (4.52)$$

$$\phi(t) = \begin{cases} \phi_1(t), & 0 \leq t < \frac{T}{2} \\ \phi_2(t), & \frac{T}{2} < t \leq T \end{cases} \quad (4.53)$$

$$\begin{aligned} \phi_1(t) &= c_{10} + c_{11}t + \frac{1}{2}c_{12}t^2 + \frac{1}{3}c_{13}t^3 \\ \phi_2(t) &= c_{20} + c_{21}t + \frac{1}{2}c_{22}t^2 + \frac{1}{3}c_{23}t^3 \end{aligned} \quad (4.54)$$

The origin of time has been shifted because a certain uniformity in some of the expressions which follow is thereby attained. We continue to take $\tau > 0$. The sketch in Fig. 5 may be helpful in determining the subdivisions of the interval of integration $[0, T - \tau]$. Referring also to Eq. 1.5, we have for $0 < \tau < T/2$,

$$\begin{aligned} \chi(\tau, \beta) &= \int_0^{(T/2)-\tau} A(t; \tau) e^{2\pi i \phi_{11}(t; \tau, \beta)} dt \\ &+ \int_{(T/2)-\tau}^{T/2} A(t; \tau) e^{2\pi i \phi_{12}(t; \tau, \beta)} dt + \int_{T/2}^{T-\tau} A(t; \tau) e^{2\pi i \phi_{22}(t; \tau, \beta)} dt \end{aligned} \quad (4.55)$$

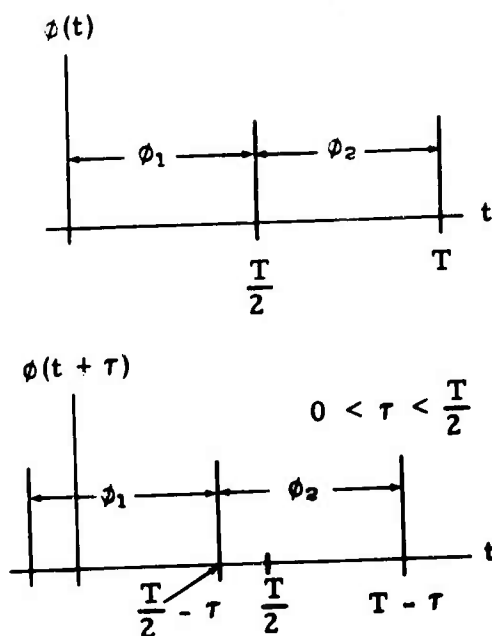


FIG. 5. Aid for Determination of Limits in Eq. 4.55.

and for $T/2 < \tau < T$,

$$\chi(\tau, \beta) = \int_0^{T-\tau} A(t; \tau) e^{2\pi i \Phi_{12}(t; \tau, \beta)} dt \quad (4.56)$$

Here, the functions Φ_{11} , etc., are given by

$$\begin{aligned} \Phi_{11}(t; \tau, \beta) &= \phi_1(t) - \phi_1(t + \tau) - \beta t \\ &= -\left[c_{11}\tau + \frac{1}{2} c_{12}(2t\tau + \tau^2) + \frac{1}{3} c_{13}(3t^2\tau + 3t\tau^2 + \tau^3) \right] - \beta t \end{aligned} \quad (4.57)$$

$$\begin{aligned} \Phi_{12}(t; \tau, \beta) &= \phi_1(t) - \phi_2(t + \tau) - \beta t = (c_{10} - c_{20}) + (c_{11} - c_{21})t \\ &\quad + \frac{1}{2} (c_{12} - c_{22})t^2 + \frac{1}{3} (c_{13} - c_{23})t^3 \\ &\quad - \left[c_{21}\tau + \frac{1}{2} c_{22}(2t\tau + \tau^2) + \frac{1}{3} c_{23}(3t^2\tau + 3t\tau^2 + \tau^3) \right] - \beta t \end{aligned} \quad (4.58)$$

$$\Phi_{22}(t; \tau, \beta) = \phi_2(t) - \phi_2(t + \tau) - \beta t$$

$$= - \left[c_{21}\tau + \frac{1}{2} c_{22}(2t\tau + \tau^2) + \frac{1}{3} c_{23}(3t^2\tau + 3t\tau^2 + \tau^3) \right] - \beta t \quad (4.59)$$

and, as before,

$$A(t; \tau) = a(t)a(t + \tau) \quad (4.60)$$

Were the calculation of $|\chi(\tau, \beta)|^2$ to be done completely, it would be necessary to take their relative phases into account when adding up the three terms of Eq. 4.55. This raises the possibility of contriving a destructive interference. However, such an endeavor would succeed only if the constituent side lobes were coincident, which is unlikely, and this course will not be pursued here. Since the functions $\Phi_{11}(t; \tau, \beta)$ and $\Phi_{22}(t; \tau, \beta)$ have the same character as the function $\Phi(t; \tau, \beta)$ previously dealt with, the side lobes contributed by the first and third integrals in Eq. 4.55 will be similar to those encountered before, except that they will extend outward from the origin only to $\tau = T/2$. The second integral in Eq. 4.55, as well as the integral in Eq. 4.56, offers the possibility of yielding a different kind of result. For each of these integrals we have, from Eq. 4.58,

$$\begin{aligned} \frac{\partial \Phi_{12}(t; \tau, \beta)}{\partial t} &= [(c_{11} - c_{21} - c_{22}\tau - c_{23}\tau^2] \\ &\quad + (c_{12} - c_{22} - 2c_{23}\tau)t + (c_{13} - c_{23}t^2) - \beta \end{aligned} \quad (4.61)$$

and

$$\frac{\partial^2 \Phi_{12}(t; \tau, \beta)}{\partial t^2} = (c_{12} - c_{22} - c_{23}\tau) + 2(c_{13} - c_{23})t \quad (4.62)$$

Starting with these two equations and proceeding as in the previous examples, we want to see whether the constants c_{ij} can be chosen in such a way that the side lobe corresponding to Eq. 4.56 is displaced outward along the β -axis beyond the maximum expected values $\pm|\beta|_m$. If it is possible to do this, then, to within the SPA, the $\tau - \beta$ plane will be effectively free of ambiguity for $\tau > T/2$. In view of the remarks above concerning interference among the terms of Eq. 4.55, we should similarly try to displace the side lobe corresponding to the second term there.

$$\int_{(T/2)-\tau}^{T/2} A(t; \tau) e^{2\pi i \Phi_{12}(t; \tau, \beta)} dt \quad (4.63)$$

This displacement of side lobes involves an additional benefit which is due to the fact that, to within the narrow-band approximation, the volume under the $|\chi(\tau, \beta)|^2$ surface is independent of the waveform and depends only upon the signal "energy" E :

$$\iint_{-\infty}^{\infty} |\chi(\tau, \beta)|^2 d\tau d\beta = \iint_{-\infty}^{\infty} |u(t)|^2 |u(t + \tau)|^2 dt d\tau \approx (2E)^2 \quad (4.64)$$

Hence, any volume moved from within to without the domain of physically interesting values of τ and β will diminish that which remains.

It turns out that the number of constants c_{ij} ($i, j = 1, 2$) can be reduced, so as to simplify the algebra, without jeopardizing the result sought. It is noted first that the c_{i0} enter only in Eq. 4.58, and that these constants will therefore play no role in the displacement of side lobes. Stated otherwise, finite discontinuities in the phase modulation $\phi(t)$ are irrelevant to the displacement of side lobes. But since a discontinuity in $\phi(t)$ is nonphysical anyway, the c_{i0} will be retained for the purpose of keeping $\phi(t)$ formally continuous. The c_{i1} will be retained for the purpose of studying the effect of a finite discontinuity in $\phi'(t)$, the instantaneous frequency, noting that such discontinuities seem to be in keeping with fundamental limitations upon the signal. Of the remaining constants, either the c_{i2} or the c_{i3} can be put equal to zero without prejudicing our objective. Under the second of these alternatives, the simpler theorems of Section 2 are not applicable because $\partial^2 \Phi_{11}(t; \tau, \beta) / \partial t^2$ and $\partial^2 \Phi_{22}(t; \tau, \beta) / \partial t^2$ vanish identically, and the side lobes associated with $\Phi_{11}(t; \tau, \beta)$ and $\Phi_{22}(t; \tau, \beta)$ manifest themselves only as straight lines, as in Eq. 4.7. So, for the present we put

$$c_{12} = c_{22} = 0 \quad (4.65)$$

The alternative is considered in the next example.

Equations 4.61 and 4.62 now read

$$\begin{aligned} \frac{\partial \Phi_{12}(t; \tau, \beta)}{\partial t} &= [(c_{11} - c_{21} - c_{23}\tau^2) \\ &\quad - 2c_{23}\tau t + (c_{13} - c_{23})t^2] - \beta = \Theta_{12}(t; \tau) - \beta \end{aligned} \quad (4.66)$$

and

$$\frac{\partial^2 \Phi_{12}(t; \tau, \beta)}{\partial t^2} = -2c_{23}\tau + 2(c_{13} - c_{23})t \quad (4.67)$$

The terminal curves associated with Eq. 4.56 are given by

$$f_{12}^{(1)}(\tau) = [\Theta_{12}(t_j; \tau)]_{t_j=0} = c_{11} - c_{21} - c_{23}\tau^2 \quad (4.68)$$

$$f_{12}^{(2)}(\tau) = [\Theta_{12}(t_j; \tau)]_{t_j=T-\tau} = c_{13}(\tau - T)^2 + (c_{11} - c_{21} - c_{23}T^2) \quad (4.69)$$

with $T/2 < \tau < T$, and those associated with Eq. 4.63 are given by

$$g_{12}^{(1)}(\tau) = [\Theta_{12}(t_j; \tau)]_{t_j=T/2-\tau} = c_{13}\left(\tau - \frac{T}{2}\right)^2 + \left(c_{11} - c_{21} - c_{23}\frac{T^2}{4}\right) \quad (4.70)$$

$$g_{12}^{(2)}(\tau) = [\Theta_{12}(t_j; \tau)]_{t_j=T/2} = -c_{23}\left(\tau + \frac{T}{2}\right)^2 + \left(c_{11} - c_{21} + c_{13}\frac{T^2}{4}\right) \quad (4.71)$$

with $0 < \tau < T/2$. These curves are sketched in Fig. 6 and 7 under the additional conditions

$$c_{23} < c_{13} < 0 \quad (4.72)$$

$$b_1 + (c_{13} - c_{23})\left(\frac{T}{2}\right)^2 \geq |\beta|, \quad (4.73)$$

where, for convenience, we have put

$$b_1 = c_{11} - c_{21} \quad (4.74)$$

The shaded regions (side lobes) are determined as before by the behavior of $[\partial^2 \Phi_{12}(t; \tau, \beta)/\partial t^2]_{t=t_j}$, Eq. 4.67, on the appropriate t - and τ -intervals.

The inequality (4.73) requires that the frequency $\phi'(t)$ be discontinuous at time $t = T/2$. To see this, suppose to the contrary that continuity obtains, $\phi'_1(T/2) = \phi'_2(T/2)$. Then

$$b_1 = c_{11} - c_{21} = -(c_{13} - c_{23})\left(\frac{T}{2}\right)^2 \quad (4.75)$$

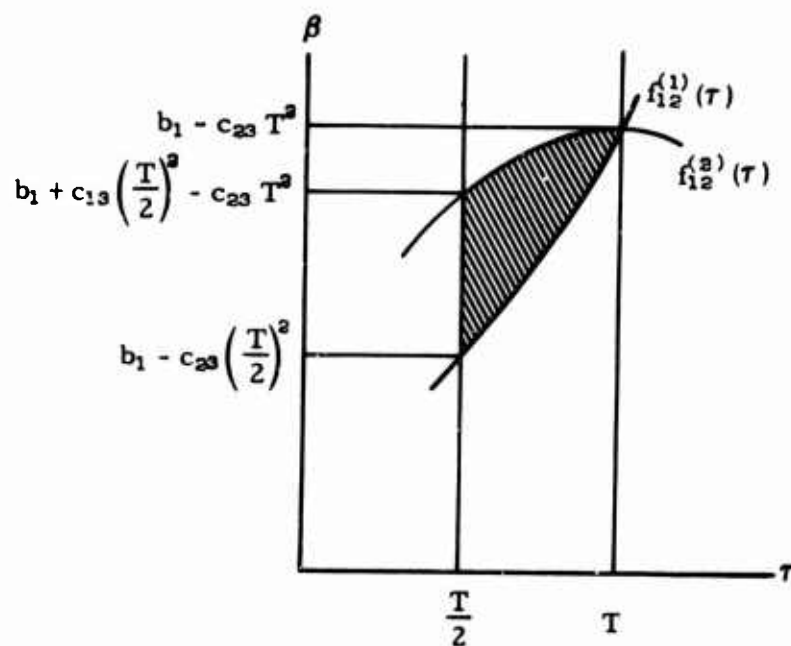


FIG. 6. Side Lobe Corresponding to Eq. 4.56.

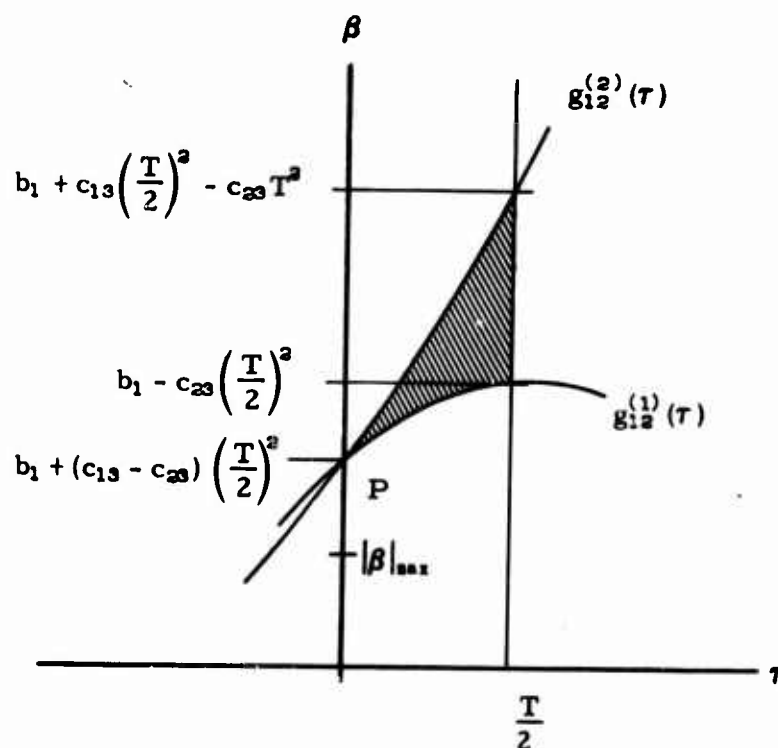


FIG. 7. Side Lobe Corresponding to Eq. 4.63.

in which case Eq. 4.73 is not satisfied, and the point P in Fig. 6 coincides with the origin. If the required jump in frequency is denoted by $(\Delta f)_{12}$,

$$(\Delta f)_{12} = \psi'_2\left(\frac{T}{2}\right) - \psi'_1\left(\frac{T}{2}\right) = -b_1 - (c_{13} - c_{23})\left(\frac{T}{2}\right)^2$$

and, upon imposing Eq. 4.73,

$$(\Delta f)_{12} \leq -|\beta|_s \quad (4.76)$$

Having seen the effect of continuity in frequency, we may now put

$$c_{11} = c_{21} = 0 \quad (4.77)$$

Upon combining previous results, we have

$$\phi(t) = \begin{cases} \phi_1(t) = c_{10} + \frac{1}{3} c_{13} t^3, & 0 \leq t \leq \frac{T}{2} \\ \phi_2(t) = c_{20} + \frac{1}{3} c_{23} t^3, & \frac{T}{2} \leq t \leq T \end{cases} \quad (4.78)$$

with c_{10} and c_{20} chosen to obtain continuity in phase at $t = T/2$, and with c_{13} and c_{23} satisfying

$$c_{23} < c_{13} < 0 \quad (4.79)$$

$$(c_{13} - c_{23})\left(\frac{T}{2}\right)^2 \geq |\beta|_s \quad (4.80)$$

The side lobes corresponding to the first and third integrals in Eq. 4.55, i. e., to Φ_{11} and Φ_{22} , are also of interest. By previous results, and under the conditions (4.79), they appear as sketched in Fig. 8. The bounding curves are again arcs of parabolas. The fact that the two lobes do not overlap is a desirable state of affairs, since the ambiguity outside the central lobe of $|\chi(\tau, \beta)|^2$ but within the region physically interesting values of τ and β tends thereby to be reduced. The complete ambiguity diagram (for $\tau > 0$, and without the central lobe) is obtained by superposition of Fig. 6 through 8, with b_1 now equal to zero.

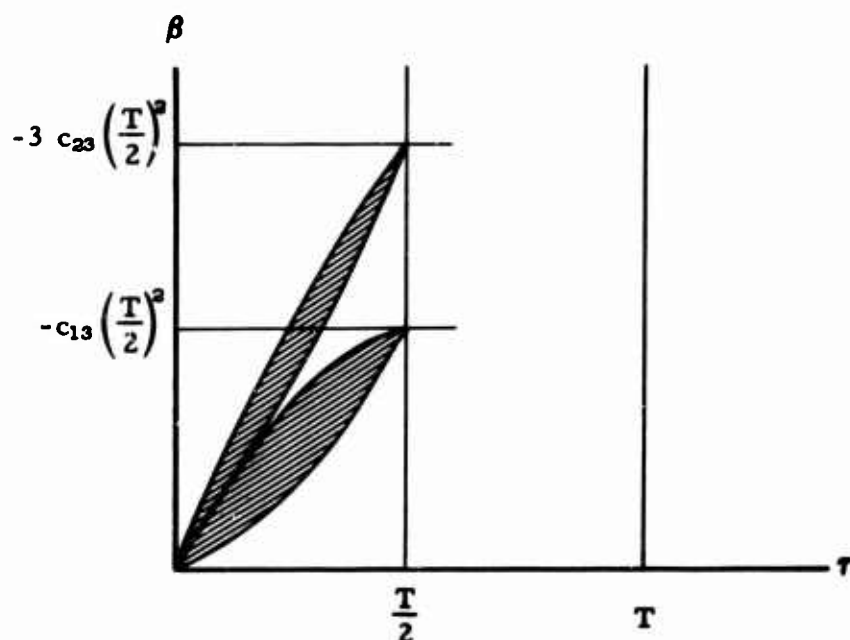


FIG. 8. Side Lobes Corresponding to First and Third Terms of Eq. 4.55.

Example 5.

The alternative (and possibly more practicable) choice of the constants c_{2j} , c_{3j} discussed just prior to Eq. 4.65 is now discussed. At the same time, we shall subdivide the signal duration T into more parts (four) with the intention of displacing more of the ambiguity volume beyond $\pm|\beta|_m$, and of distributing in azimuth those side lobes that are connected to the origin. Thus, we have

$$\phi(t) = \begin{cases} \phi_1(t) = c_{10} + \frac{1}{2} c_{12} t^2, & 0 \leq t \leq \frac{T}{4} \\ \phi_2(t) = c_{20} + \frac{1}{2} c_{22} t^2, & \frac{T}{4} \leq t \leq \frac{T}{2} \\ \phi_3(t) = c_{30} + \frac{1}{2} c_{32} t^2, & \frac{T}{2} \leq t \leq \frac{3T}{4} \\ \phi_4(t) = c_{40} + \frac{1}{2} c_{42} t^2, & \frac{3T}{4} \leq t \leq T \end{cases} \quad (4.81)$$

the constants c_{i0} being chosen so as to make the phase modulation $\phi(t)$ continuous. This $\phi(t)$ leads to the functions

$$\Phi_{ij}(t; \tau, \beta) = \phi_i(t) - \phi_j(t + \tau) - \beta t \quad (4.82)$$

$$\frac{\partial \Phi_{ij}(t; \tau, \beta)}{\partial t} = \phi'_i(t) - \phi'_j(t + \tau) - \beta = c_{i2}t - c_{j2}(t + \tau) - \beta \quad (4.83)$$

$$\Theta_{ij}(t; \tau) = c_{i2}t - c_{j2}(t + \tau) \quad (4.84)$$

and

$$\frac{\partial^2 \Phi_{ij}(t; \tau, \beta)}{\partial t^2} = c_{i2} - c_{j2} \quad (4.85)$$

where $i \leq j$ and $i, j = 1, 2, 3, 4$. The calculations divide naturally into four groups corresponding to the τ -intervals,

$$(i - 1) \frac{T}{4} \leq \tau \leq i \frac{T}{4} \quad (4.86)$$

of which only the first is considered in detail.

The interval $0 < \tau \leq T/4$ involves the following phase functions and limits:

	t_1	t_2		t_1	t_2
Φ_{11}	0	$\frac{T}{4} - \tau$	Φ_{12}	$\frac{T}{4} - \tau$	$\frac{T}{4}$
Φ_{22}	$\frac{T}{4}$	$\frac{T}{2} - \tau$	Φ_{23}	$\frac{T}{2} - \tau$	$\frac{T}{2}$
Φ_{33}	$\frac{T}{2}$	$\frac{3T}{4} - \tau$	Φ_{34}	$\frac{3T}{4} - \tau$	$\frac{3T}{4}$
Φ_{44}	$\frac{3T}{4}$	$T - \tau$			

(4.87)

The contribution of each of these to $\chi(\tau, \beta)$ is given by the integral

$$\int_{t_1}^{t_2} A(t; \tau) e^{2\pi i \Phi_{ij}(t; \tau, \beta)} dt \quad (4.88)$$

The correctness of (4.87) and (4.88) may be verified by means of a sketch similar to Fig. 5.

The phase functions Φ_{ij} correspond to linear FM and lead to the degenerate side lobes,

$$\beta = -c_{12}T \quad (4.89)$$

(see Eq. 4.7 and the attendant discussion.) It turns out that we can choose two of the c_{12} to be negative (and unequal) and two to be positive (and unequal). By so doing, the ambiguity volume near the origin will tend to be distributed in azimuth, rather than concentrated somewhere so as to result in a prominent ridge or peak.

The side lobes corresponding to the phase functions Φ_{ij} , $i < j$, can, at the same time, be displaced outward beyond $\pm|\beta|_m$. The terminal curves are given by

$$\begin{aligned} f_{1j}^{(1)}(\tau) &= [\Theta_{1j}(t; \tau)]_{t=T/4-\tau} = -c_{12}\tau + (c_{12} - c_{22}) \frac{iT}{4} \\ f_{1j}^{(2)}(\tau) &= [\Theta_{1j}(t; \tau)]_{t=T/4} = -c_{22}\tau + (c_{12} - c_{22}) \frac{iT}{4} \end{aligned} \quad (4.90)$$

Under the conditions

$$c_{12} < 0, \quad c_{22} < 0, \quad c_{32} > 0, \quad c_{42} > 0 \quad (4.91)$$

$$(c_{12} - c_{22}) \frac{T}{4} \geq |\beta|_m \quad (4.92)$$

and

$$(c_{42} - c_{32}) \frac{3T}{4} \geq |\beta|_m \quad (4.93)$$

Eq. 4.85, 4.89, and 4.90 yield the partial ambiguity diagram sketched in Fig. 9. The dashed lines are the loci of Eq. 4.89; the dotted lines represent the continuation of the displaced side lobes into the next interval, $T/4 < \tau < T/2$.

The remaining τ -intervals involve equations similar to those considered above. Only one additional condition,

$$c_{32} \frac{T}{2} \geq |\beta|_m \quad (4.94)$$

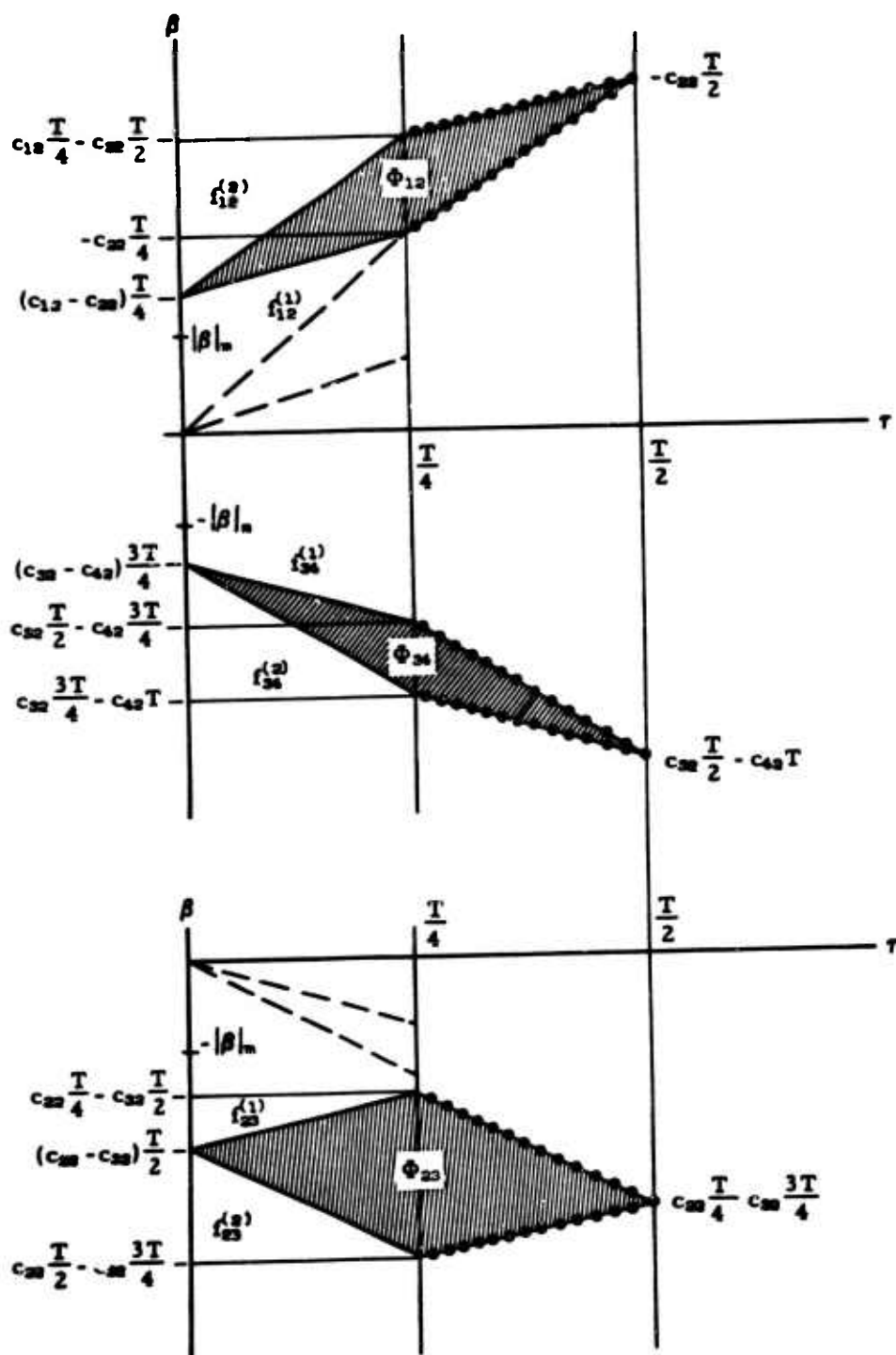


FIG. 9. Partial Ambiguity Diagram for the Modulation (4.81) Under the Conditions (4.91) Through (4.93). The remaining side lobes lie below $-|\beta|_m$.

is required to insure that the remaining side lobes are outside the range of physically interesting values of β . There are three, they all lie below $-\beta_m$, and each has the character of the lowermost side lobe shown in Fig. 9, except that it is displaced to the right by one or two multiples of $T/4$. The area of the displaced side lobe Φ_{ij} , $i < j$, is $|c_{i2} - c_{j2}|(T/4)^2$.

For the case of an $a(t)$ which is nearly rectangular and normalized to unity, $A(t_j; \tau) \approx 1/T$, and the displaced ambiguity volume computed according to Eq. 3.17 under the assumption that the side lobes are disjoint is easily found to be $3/8$ of the total (unity). If the number N of subdivisions of the signal duration T is decreased to 2, the displaced volume is $1/4$; if N is increased to 8, it is $7/16$; and if N is increased indefinitely, it tends toward $1/2$. It turns out that the fraction of the displaced side-lobe area which lies to the left of $\tau = T/2$ increases as N increases. Hence, an $A(t_j; \tau)$ which decreased with increasing τ would result in a greater displaced volume.

The question of whether other phase modulations will yield results which are more favorable in the sense of displaced ambiguity volume has not been pursued.

Example 6.

Although the finite jumps in the instantaneous frequency $\phi'(t)$ which were involved in the previous two examples do not seem to violate any fundamental restrictions upon the signal, they are likely to be difficult to realize in practice. It is therefore proper to ascertain the consequences of replacing the jumps by continuous, rapid transitions. To this end, we consider the modulations

$$\phi_a(t) = \begin{cases} \phi_1(t) = c_{10} + \frac{1}{2} c_{12} t^2, & 0 \leq t \leq \frac{T}{2} \\ \phi_3(t) = c_{30} + \frac{1}{2} c_{32} t^2, & \frac{T}{2} \leq t \leq T \end{cases} \quad (4.95)$$

which will be made to have a frequency jump at $t = T/2$, together with the modulation

$$\phi_b(t) = \begin{cases} \phi_1(t) = c_{10} + \frac{1}{2} c_{12} t^2, & 0 \leq t \leq \frac{T}{2} - \epsilon \\ \phi_2(t) = c_{20} + c_{21} t + \frac{1}{2} c_{22} t^2, & \frac{T}{2} - \epsilon \leq t \leq \frac{T}{2} + \epsilon \\ \phi_3(t) = c_{30} + \frac{1}{2} c_{32} t^2, & \frac{T}{2} + \epsilon \leq t \leq T \end{cases} \quad (4.96)$$

for which the instantaneous frequency will be made continuous by virtue of a linear transition during the interval, $T/2 - \epsilon \leq t \leq T/2 + \epsilon$, with

$$0 < \frac{\epsilon}{T} \ll 1 \quad (4.97)$$

The constants c_{i0} are, again, supposed to be chosen so that $\phi_a(t)$ and $\phi_b(t)$ are continuous. In the case of $\phi_b(t)$, the imposition of continuity in frequency yields

$$\begin{aligned} c_{21} &= -\frac{1}{2\epsilon} \left(\frac{T}{2} + \epsilon \right) \left(\frac{T}{2} - \epsilon \right) (c_{32} - c_{12}) \\ c_{22} &= \frac{1}{2\epsilon} \left[c_{32} \left(\frac{T}{2} + \epsilon \right) - c_{12} \left(\frac{T}{2} - \epsilon \right) \right] \end{aligned} \quad (4.98)$$

The constants c_{12} , c_{32} are chosen so that the degenerate (linear FM) side lobes corresponding to ϕ_{11} and ϕ_{33} have opposite slopes, and the remaining side lobe for $\phi_a(t)$ is displaced beyond $+\beta|_m$:

$$c_{12} > 0, \quad c_{32} < 0, \quad -c_{32} \frac{T}{2} \geq |\beta|_m \quad (4.99)$$

For definiteness, we also take $|c_{12}| < |c_{32}|$. The resulting ambiguity diagram for $\phi_a(t)$ is sketched in Fig. 10; as before, the linear FM side lobes are indicated by the dashed lines.

The labeling, $f_{ij}^{(k)}(\tau)$ and $g_{ij}^{(k)}(\tau)$, of the terminal curves employed in this example means that the associated phase function is ϕ_{ij} , with $i < j$, and that this is the k^{th} time ϕ_{ij} has occurred as $\tau \rightarrow T$ from $\tau = 0^+$. The functions $f_{ij}^{(k)}(\tau)$ correspond to the lower, and the functions $g_{ij}^{(k)}(\tau)$ to the upper, limits of the integrals with which they are connected.

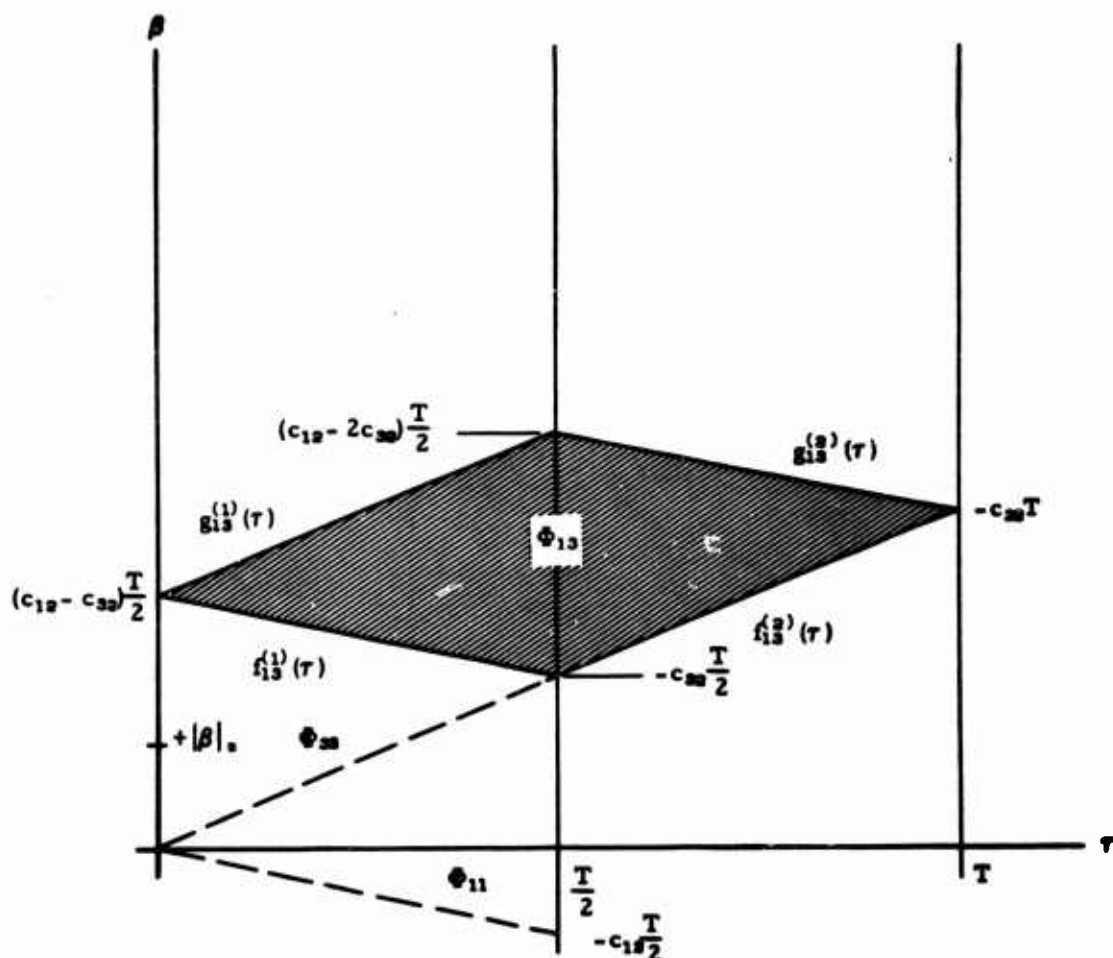
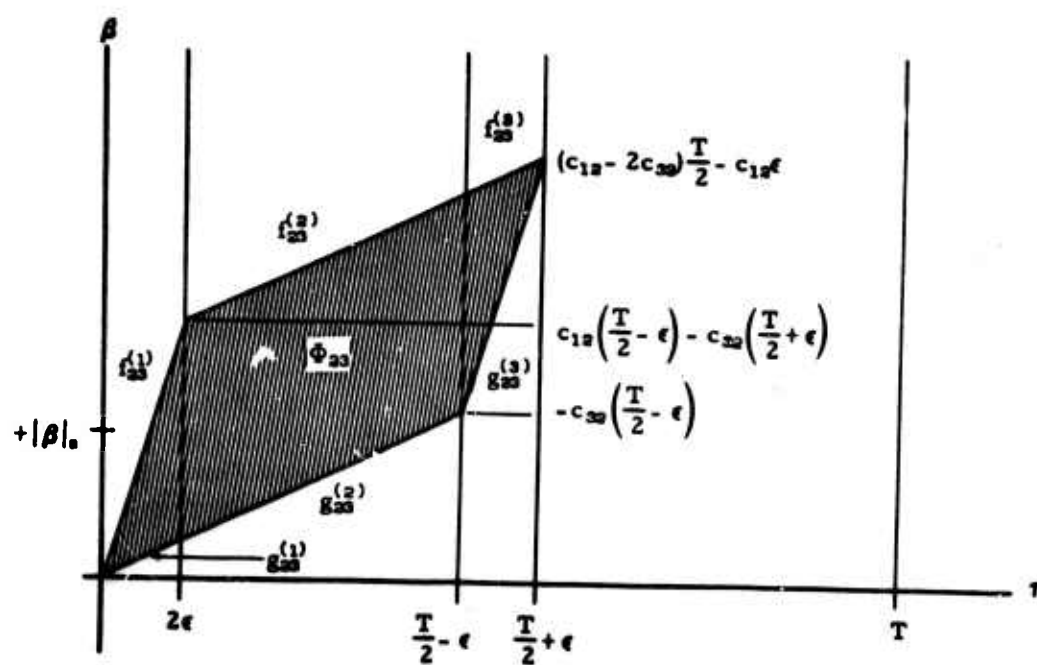
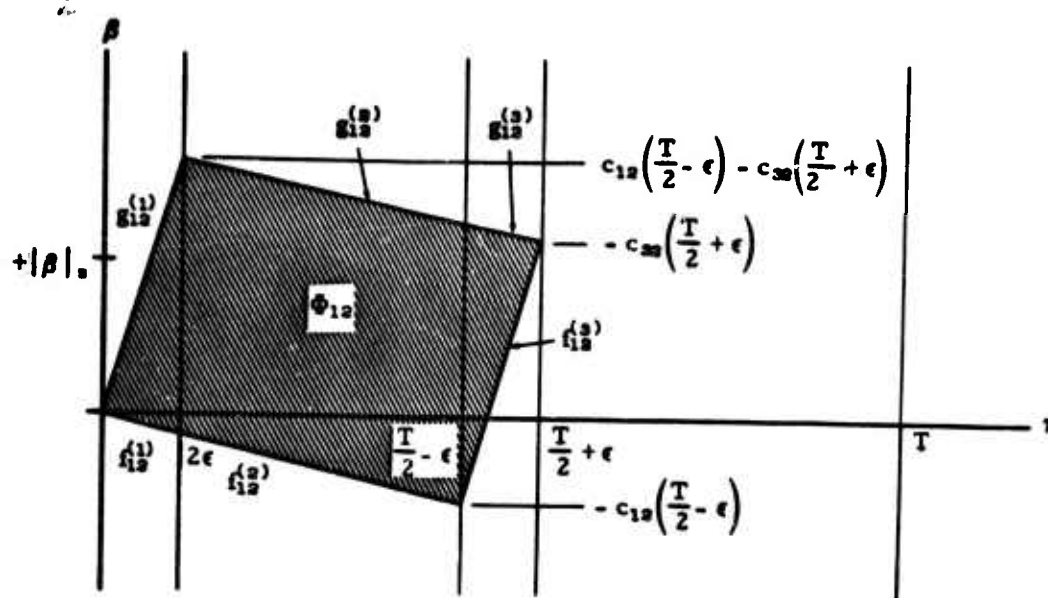


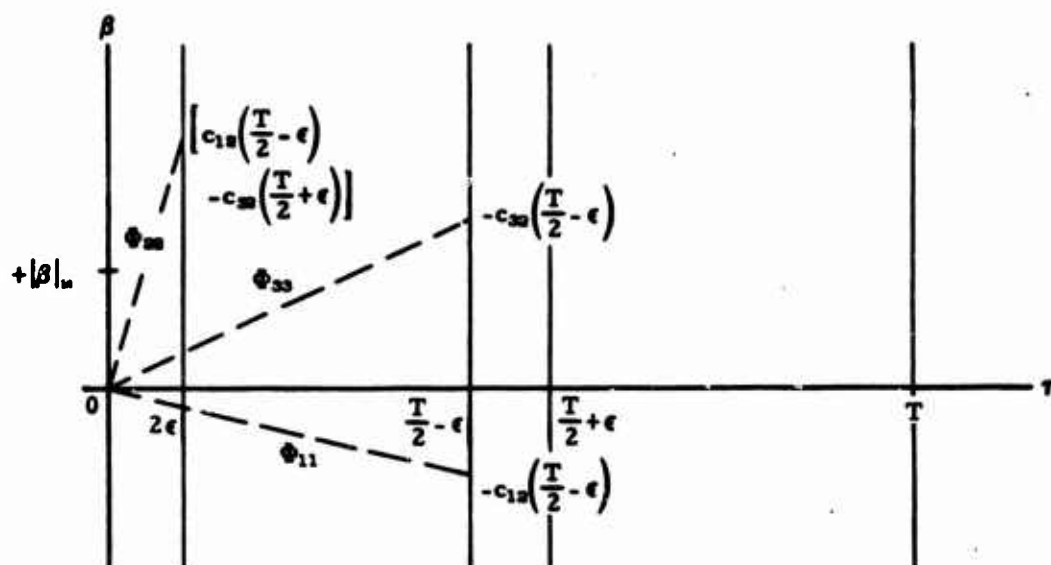
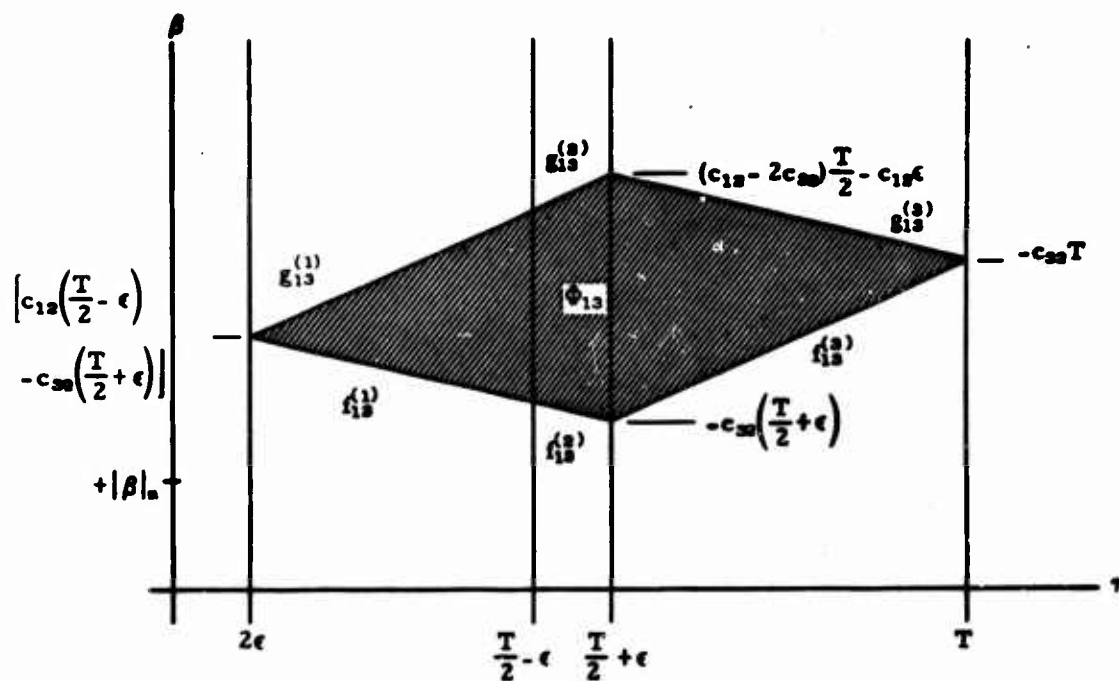
FIG. 10. Ambiguity Diagram for the Modulation $\phi_b(t)$, Eq. 4.95, and the Conditions (4.99).

With the conditions (4.99) supplemented by Eq. 4.98, we obtain for $\phi_b(t)$ the side lobes sketched in Parts a and b of Fig. 11. Comparison of this figure with Fig. 10 shows that the replacing of the frequency jump at $t = T/2$ by a linear transition has had the undesirable result of generating additional side lobes (the linear FM lobe Φ_{22} in Part b of Fig. 11, and both of the side lobes in the two parts of Fig. 11), portions of which lie within the domain of expected values of τ and β . However, under Eq. 4.97, and to within the SPA, contributions to $|\chi(\tau, \beta)|$ of these additional side lobes will be small compared to $|\chi(0, 0)|$, as will now be shown.



(Part a)

FIG. 11. Side Lobes for the Modulation $\phi_b(t)$, Eq. 4.96, and the Conditions (4.98) and (4.99).



(Part b)

FIG. 11. (Contd.)

For the side lobes Φ_{12} and Φ_{22} we have, using Eq. 4.98,

$$\frac{\partial^2 \Phi_{12}(t; \tau, \beta)}{\partial t^2} = c_{12} - c_{22} = \frac{1}{2\epsilon} \left(\frac{T}{2} + \epsilon \right) (c_{12} - c_{22})$$

$$\frac{\partial^2 \Phi_{22}(t; \tau, \beta)}{\partial t^2} = c_{22} - c_{32} = \frac{1}{2\epsilon} \left(\frac{T}{2} - \epsilon \right) (c_{22} - c_{12})$$

and by Eq. 3.17,

$$|\chi_{12}(\tau, \beta)| \approx \frac{(2\epsilon)^{1/2}}{\left(\frac{T}{2} + \epsilon \right)^{1/2} (c_{12} - c_{22})^{1/2}} A(t_j^{(12)}; \tau) \quad (4.100)$$

$$|\chi_{22}(\tau, \beta)| \approx \frac{(2\epsilon)^{1/2}}{\left(\frac{T}{2} - \epsilon \right)^{1/2} (c_{12} - c_{22})^{1/2}} A(t_j^{(22)}; \tau) \quad (4.101)$$

where the stationary points are given by

$$t_j^{(12)} = \frac{\left(\frac{T}{2} + \epsilon \right) \left(\frac{T}{2} - \epsilon \right) (c_{12} - c_{22}) + \left[c_{22} \left(\frac{T}{2} + \epsilon \right) - c_{12} \left(\frac{T}{2} - \epsilon \right) \right] \tau + 2\beta\epsilon}{\left(\frac{T}{2} + \epsilon \right) (c_{12} - c_{22})}$$

$$= \frac{T}{2} - \tau + 0 \left(\frac{\epsilon}{T} \right), \quad 0 < \tau \leq \frac{T}{2} + \epsilon \quad (4.102)$$

and

$$t_j^{(22)} = \frac{\left(\frac{T}{2} + \epsilon \right) \left(\frac{T}{2} - \epsilon \right) (c_{22} - c_{12}) + (c_{22}\tau + \beta)(2\epsilon)}{\left(\frac{T}{2} - \epsilon \right) (c_{22} - c_{12})}$$

$$= \frac{T}{2} + 0 \left(\frac{\epsilon}{T} \right), \quad 0 < \tau \leq \frac{T}{2} + \epsilon \quad (4.103)$$

For the displaced side lobe Φ_{13} , we have

$$\frac{\partial^2 \Phi_{13}(t; \tau, \beta)}{\partial t^2} = c_{12} - c_{32}$$

$$|\chi_{13}(\tau', \beta)| \approx \frac{1}{(c_{12} - c_{32})^{1/2}} A(t_j^{(13)}; \tau') \quad (4.104)$$

and

$$t_j^{(13)} = \frac{c_{32}\tau' + \beta}{c_{12} - c_{32}}, \quad 2\epsilon \leq \tau' \leq T \quad (4.105)$$

From Part b of Fig. 11 we find (by requiring the ordinate β to lie within the shaded region Φ_{13}) that

$$\frac{T}{2} - \tau' + 0\left(\frac{\epsilon}{T}\right) \leq t_j^{(13)} \leq \frac{T}{2} + 0\left(\frac{\epsilon}{T}\right) \quad \text{for } 2\epsilon \leq \tau' \leq \frac{T}{2} + \epsilon \quad (4.106)$$

and that

$$0 \leq t_j^{(13)} \leq T - \tau' \quad \text{for } \frac{T}{2} + \epsilon \leq \tau' \leq T \quad (4.107)$$

It is now shown that if τ and $t_j^{(12)}$ are prescribed according to Eq. 4.102, then a $t_j^{(13)}$ and a τ' satisfying Eq. 4.106 can be found such that

$$A(t_j^{(12)}; \tau) = A(t_j^{(13)}; \tau') \quad (4.108)$$

We use the continuity of $a(t)$ and work only to within terms which are $O(\epsilon/T)$. By doing the latter, small areas of the side lobes are excluded; these can be taken into account by a continuity argument. We have

$$A(t_j^{(12)}; \tau) = a(t_j^{(12)})a(t_j^{(12)} + \tau) \approx a\left(\frac{T}{2} - \tau\right)a\left(\frac{T}{2}\right), \quad 0 < \tau < \frac{T}{2}$$

Since $a(t)$ is continuous and vanishes for $t > T$, there exists a $\tilde{\tau}$, $0 < \tilde{\tau} < T/2$, such that

$$A(t_j^{(12)}; \tau) = a\left(\frac{T}{2} + \tilde{\tau}\right)a\left(\frac{T}{2}\right), \quad 0 < \tilde{\tau} < \frac{T}{2} \quad (4.109)$$

Now, Eq. 4.106 is satisfied by $t_j^{(13)} = T/2$ and $\tau' = \tilde{\tau}$, and for these values we obtain Eq. 4.108. The function $A(t_j^{(23)}; \tau)$, where $t_j^{(23)}$ and τ satisfy Eq. 4.103, is given by

$$A(t_j^{(23)}; \tau) = a(t_j^{(23)})a(t_j^{(23)} + \tau) \approx a\left(\frac{T}{2}\right)a\left(\frac{T}{2} + \tau\right), \quad 0 < \tau < \frac{T}{2}$$

Upon comparing this with Eq. 4.109, it is seen that

$$A(t_j^{(23)}; \tau) = A(t_j^{(13)}; \tau') \quad (4.110)$$

for $t_j^{(13)} = T/2$ and $\tau' = \tau$. Next, the approximations in Eq. 4.108 and 4.110 are used in Eq. 4.100 and 4.101, thus arriving at

$$|\chi_{12}(\tau, \beta)| \approx 2 \left(\frac{\epsilon}{T} \right)^{1/2} |\chi_{13}(\tau', \beta')|$$

$$|\chi_{23}(\tau, \beta)| \approx 2 \left(\frac{\epsilon}{T} \right)^{1/2} |\chi_{13}(\tau', \beta')|$$

by way of Eq. 4.104. Finally, we employ the general and exact relationship,

$$|\chi(\tau', \beta')| \leq |\chi(0, 0)| \quad (4.111)$$

to obtain

$$|\chi_{12}(\tau, \beta)| \lesssim 2 \left(\frac{\epsilon}{T} \right)^{1/2} |\chi(0, 0)|$$

$$|\chi_{13}(\tau, \beta)| \lesssim 2 \left(\frac{\epsilon}{T} \right)^{1/2} |\chi(0, 0)| \quad (4.112)$$

as claimed.

The demonstration that $|\chi_{22}(\tau, \beta)| \ll |\chi(0, 0)|$ follows along similar lines. Application of Erdelyi's theorem to

$$\chi_{11}(\tau', \beta') = \int_0^{T/2 - \epsilon - \tau'} A(t; \tau') e^{2\pi i \Phi_{11}(t; \tau', \beta')} dt$$

and

$$\chi_{22}(\tau, \beta) = \int_{T/2 - \epsilon}^{T/2 + \epsilon - \tau'} A(t; \tau) e^{2\pi i \Phi_{22}(t; \tau, \beta)} dt$$

gives

$$|\chi_{11}(\tau', \beta')| = \frac{a \left(\frac{T}{2} - \epsilon - \tau' \right) a \left(\frac{T}{2} - \epsilon \right)}{2\pi |c_{12}\tau' + \beta'|} + O \left[\frac{1}{(2\pi c_{12})^2} \right]$$

and

$$|X_{22}(\tau, \beta)| = \frac{1}{2\pi |c_{22}\tau + \beta|} \left\{ A^2 \left(\frac{T}{2} + \epsilon - \tau; \tau \right) + A^2 \left(\frac{T}{2} - \epsilon; \tau \right) - 2A \left(\frac{T}{2} + \epsilon - \tau; \tau \right) A \left(\frac{T}{2} - \epsilon; \tau \right) \cos 2\pi (c_{22}\tau + \beta)(2\epsilon - \tau) \right\}^{\frac{1}{2}} + O \left[\frac{1}{(2\pi c_{22})^2} \right]$$

where

$$0 < \tau' \leq \frac{T}{2} - \epsilon, \beta' \neq c_{12}\tau' \quad (4.113)$$

and

$$0 < \tau < 2\epsilon, \beta \neq c_{22}\tau \quad (4.114)$$

Upon using Eq. 4.97, 4.114, and the second of Eq. 4.98, we obtain

$$|X_{22}(\tau, \beta)| \approx \frac{\epsilon}{T} \frac{4a^2 \left(\frac{T}{2} \right)}{\pi | (c_{33} - c_{12})\tau + \frac{4\epsilon}{T} \beta |} |\sin \pi (c_{22}\tau + \beta)(2\epsilon - \tau)|$$

$$|X_{22}(\tau, \beta)| \leq \frac{\epsilon}{T} \frac{4a^2 \left(\frac{T}{2} \right)}{\pi | (c_{33} - c_{12})\tau + \frac{4\epsilon}{T} \beta |}$$

and

$$\frac{|X_{22}(\tau, \beta)|}{|X_{11}(\tau', \beta')|} \leq \frac{\epsilon}{T} \left\{ \frac{8a \left(\frac{T}{2} \right) |c_{12}\tau' + \beta'|}{a \left(\frac{T}{2} - \tau' \right) | (c_{33} - c_{12})\tau + \frac{4\epsilon}{T} \beta |} \right\}$$

Given τ and β , it is always possible to find a τ' and a β' satisfying (4.113) such that the quantity within the braces is unity. Hence, by means of Eq. 4.111, we arrive at

$$|X_{22}(\tau, \beta)| \leq \frac{|X_{22}(\tau, \beta)|}{|X_{11}(\tau', \beta')|} |X(0, 0)| \leq \frac{\epsilon}{T} |X(0, 0)| \quad (4.115)$$

as was to be shown.

Phase modulations of the sort considered in this and the previous example have also been discussed, in a different way, by Rihaczek and Mitchell (Ref. 18).

5. SUMMARY AND DISCUSSION OF RESULTS

It has been shown that the stationary phase approximation (SPA) of Woodward's time-frequency correlation function $\chi(\tau, \beta)$, Eq. 1.1, can be elaborated upon so as to yield inequalities (or, alternatively, families of curves) specifying those domains of the time-frequency ($\tau - \beta$) plane on which the ambiguity function $|\chi(\tau, \beta)|^2$ is relatively large (side lobes). The elaboration consisted of working a condition satisfied by the stationary points for large $|\chi(\tau, \beta)|^2$ into one satisfied by β and τ . A brief, general discussion was given in Section 3, and some examples were worked in Section 4. The method fails in some small neighborhood of the origin ($\tau = 0 = \beta$), but the behavior of $|\chi(\tau, \beta)|^2$ there can, in principle, be determined by means of a Taylor series expansion. Although the SPA theorems which are usually used do not apply along the β -axis, a more general theorem given by Erdelyi (Ref. 16 and Section 2) can be used to study the ambiguity function there; alternatively, Eq. 3.24 can be used. For modulations such that the amplitude modulation $a(t)$ varies slowly compared to the phase modulation $\phi(t)$ —except, possibly, at the beginning and end of the pulse—this technique should be a useful tool for signal design.

In the examples, we considered first those phase modulations which satisfy the hypothesis of the SPA theorems usually employed (Ref. 15 and Section 2): i. e., functions $\phi(t)$ which are analytic over the duration T of the signal. The same was assumed concerning the amplitude modulation $a(t)$. The weaker hypothesis of Erdelyi's theorem was thus satisfied at the same time. It was found that in these cases there were two side lobes (one obtainable from the other by inversion in the origin) connected to the origin and extending outward to $\tau = \pm T$.

We then considered phase modulations $\phi(t)$ in the form of polynomials which satisfied the theorems only piecewise, with $a(t)$ the same as before, and it was found that (1) some of the side lobes could be displaced outward along the β -axis beyond the largest expected values of β , $\pm|\beta|_m$, and that (2) the remaining side lobes could be distributed in azimuth about the origin. In connection with Item 1, finite discontinuities in $\phi(t)$ were found to be irrelevant, and finite discontinuities in $\phi'(t)$, the instantaneous frequency, were found to be essential. Because of the volume invariance property of $|\chi(\tau, \beta)|^2$, Eq. 4.64, signals of this sort raise the hope of approximating a qualitatively ideal ambiguity function, i. e., one having a sharply peaked central lobe and very low values over the remainder of the rectangle bounding the

expected values of τ and β . Although the existence of finite jumps in the instantaneous frequency $\phi'(t)$ does not seem to stand in contradiction to fundamental restrictions upon the signal, it is possible that they are difficult to realize in practice. For this reason, we considered in the last example a phase modulation such that the jump was replaced by a linear transition occurring in an interval ϵ . We found that the contributions to $|\chi(\tau, \beta)|^2$ of the additional side lobes generated by the transition were small compared to $|\chi(0, 0)|^2$, provided that $\epsilon \ll T$.

In cases in which a large portion of the ambiguity volume corresponding to a given signal already lies beyond $\pm|\beta|_m$, not much will be gained by the techniques of Examples 4, 5, and 6 unless the signal duration T is divided into many (equal) parts. The result of division will be a signal which is, perhaps, undesirably complicated, or which has too large a bandwidth. Let us recall that Eq. 1.1 rests upon the narrow-band approximation, and that this approximation may be a poor one for modulations such that the SPA is valid.

Finally, it is pointed out that specific modulations that are arrived at by means of the methods described here are best considered as candidate signals, because we have made use, both explicitly and implicitly, of just the leading term of an asymptotic expansion. While asymptotic expansions can yield very accurate results, it is necessary to verify their accuracy by looking at more terms in Eq. 2.7 and 2.8. Alternatively, one can resort to numerical integration of Eq. 1.1, or, possibly, to analog simulation of the matched filter which is characterized by $\chi(\tau, \beta)$.

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13. ABSTRACT <p>The stationary phase approximation of Woodward's time-frequency correlation function $X(\tau, \beta)$ is considered. It is shown, from the condition that the stationary point(s) lie within the interval of integration, that there can be developed inequalities which specify those domains of the $\tau - \beta$ plane, except for some small neighborhood of the origin, on which the ambiguity function $X(\tau, \beta) ^2$ is relatively large. Some simple phase modulations $\phi(t)$ are studied to illustrate the method and to serve as a basis for determining heuristically that appropriately placed finite discontinuities in the instantaneous frequency $\phi'(t)$ result in the displacement of a part of the ambiguity volume from within to beyond physically interesting values of β. Finite discontinuities in the phase modulation itself do not matter in this respect, and the replacing of a finite jump in $\phi'(t)$ by a linear transition occurring in a sufficiently small interval is shown to have a negligible effect. Further, the undisplaced volume can be spread out in azimuth about the origin. It appears possible to approximate an ideal ambiguity function by thus distributing the total ambiguity volume. The detailed calculations necessary for complete verification of this are not included.</p>			

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